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Galin's problem for a periodic system of stamps with friction and adhesion

Y.A. Antipov*

Department of Mathematical Sciences, University of Bath, Claverton Down, Bath BA2 7AY, UK

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Abstract

A periodic system of plane stamps is pressed onto an elastic half-plane by a central vertical force P applied to each stamp. The contact area for each stamp is divided into an inner adhesive region and two outer slipping regions, where Coulomb's law of dry friction applies. The system of singular integral equations on two different segments, which corresponds to the problem, is equivalent to a Wiener–Hopf equation for a two-components vector, for which an analytical constructive solution is obtained. Effective formulae for numerical computations for the contact stresses are presented. The effect of friction and of the distance between the stamps on the length of sliding zones is investigated. © 2000 Elsevier Science Ltd. All rights reserved.

Nomenclature

x, y	two-dimensional Cartesian coordinates
u, v	displacements
σ_y, τ_{xy}	stresses
$\sigma(x)$, $\tau(x)$	contact stresses
P	applied force
E	Young's modulus
ν	Poisson's ratio
a	half-length of a stamp
$2(c-a)$	distance between stamps
b	half-length of the adhesion zone

* Corresponding author. Tel.: 00 44 01225 826 106; fax: 00 44 01225 826 492.
E-mail address: y.antipov@maths.bath.ac.uk (Y.A. Antipov)

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μ	coefficient of friction
$U(x, y)$	Airy stress function
$\kappa = 3 - 4\nu$	Muskhelishvili's constant
$\kappa_+, \kappa_-, \beta$	parameters
$l_+(x, \xi), l_-(x, \xi)$	Hilbert's kernels
$\tau_1(x)$	auxiliary function
$h^{\alpha_0,\alpha_1}(0, b), h^{\alpha_1,\alpha_2}(b, a)$	classes of solution
$A_1, A_2, A_0, B_0, A_1^{\pm}, B_1$	constants
$c_0, c_1, c_2, \delta = \delta_1 + i\delta_2$	parameters
$\phi^*(\xi)$	Hölder function
$Y^{(1)}, Y^{(2)}, \Omega_j(x)\ (j = 0, \ldots, 3)$	analytic functions
$\alpha_0, \alpha_1, \alpha_2, \alpha, a_0, b_0, \gamma$	parameters
$\chi_1(\tau), \chi_2(\tau)$	solution of the system of integral equations with Cauchy kernels
$\chi_{j-}(\tau), \chi_{j+}(\tau)\ (j = 1, 2)$	auxiliary ('ones-sides') functions
$\Phi_j^-(s), \Phi_j^+(s)\ (j = 1, 2)$	solution of the Wiener–Hopf problem
Γ	contour of the Wiener–Hopf problem
$\mathbf{G}(s)$	matrix coefficient of the Wiener–Hopf problem
$K_0(s), K_1(s)$	elements of the matrix $\mathbf{G}(s)$
$\mathbf{f}(s), F(s)$	known vector and function
$\lambda < 1, \gamma_*, \kappa_0$	parameters
$\Gamma(s)$	Gamma function
$K_0^+(s), K_0^-(s)$	factors in the splitting of the function $K_0(s)$
$D^{\pm}$	half-planes
$\Omega_j^+(s), \Omega_j^-(s)\ (j = 1, 2)$	analytic functions in D^+, D^-
$C_j^{\pm}, D_j^{\pm}, E_j^{\pm}$	known coefficients
$\Phi_0^{\pm}(s), \Psi_1^-(s), \Sigma_{1\pm}(s), \Sigma_{2\pm}(s)$	auxiliary functions
$A_j^{\pm}, B_j$	unknown residues
C	arbitrary constant
$\Delta_{0n}^+, \Delta_{1n}^+, \Delta_n^-$	coefficients
$A_{nj}^{\pm}, B_{nj}$	solution of the infinite linear algebraic system
$f_{nj}^{\pm}, f_{nj}$	coefficients
$a_{nkj}^{\pm}, b_{nkj}$	coefficients of the expansion on λ of $A_{nj}^{\pm}, B_{nj}$
N_b, K_a, L_a	stress intensity factors
$R_1(t), R_2(t)$	auxiliary functions
$\Gamma_n, \Gamma_{0n}, \Gamma_{1n}, \Gamma_{2n}$	coefficients
$I_j(t), I_j^{(k)}(t),\ (j, k = 1, 2)$	integrals
$F(a, b; c; x)$	Gaussian hypergeometric function
$F_{kj}(t), \mathfrak{F}(\nu; t), \mathfrak{F}^*(\nu; t)$	series

1. Introduction

The fact that the stresses σ_y, τ_{xy} change their sign in the vicinity of the point of the boundary between the conditions of adhesion ($u = v = 0$) and a free boundary ($\sigma_y = \tau_{xy}$) was defined first by Abramov (1937). The investigations of Galin (1945) and Fal'kovich (1945) were among the first publications, where the mixed boundary conditions in the contact region were analysed. Fal'kovich proposed to divide the contact zone into three sections in order to prevent the stresses from oscillating. The central one is the adhesion zone and the two outer edge ones are the slipping zones without friction. Fal'kovich reduced this problem to an integrable case of a Fuch's differential equation and obtained exact formulae for the contact stresses. It was found that the characteristic equation which determines the length of the slide zone has a denumerable set of roots. Fal'kovich rejected the first root leading to Sadowsky's solution (Sadowsky, 1928)

$$\sigma_y = -\frac{P}{\pi\sqrt{a^2 - x^2}}, \quad \tau_{xy} = 0 \tag{1.1}$$

and found a different solution defined by the second root of the characteristic equation. In this case the tangential stresses have variable signs only at zero and the length of the slipping zones is approximately equal to $0.003a$. Fal'kovich's solution varies monotonically in the neighbourhood of the ends, but the normal stresses have variable signs in the adhesion region. This means that neglecting friction in the sliding zone in the appropriate contact problems is not correct.

Galin's solution (Galin, 1945) does not have this deficiency. The contact area is divided into an intermediate zone of adhesion and two zones of slippage where dry friction applies: $|\tau_{xy}| = \mu|\sigma_y|$. Galin constructed the solution by the approximate method based on conformal mapping of a region closed to the given one to the upper half-plane and the reduction to two Hilbert problems. Galin's solution becomes Sadowsky's solution (1.1) but not Fal'kovich's solution as the coefficient of friction μ tends to zero.

Another situation is the problem of an interface crack (Antipov, 1995): the boundary conditions do not degenerate as $\mu \to 0$ and the corresponding solution becomes Comninou's solution (Comninou, 1977) but not the solution on a sliding crack. This essential difference occurs because in the punch problem, the length of the slipping zone is determined from the boundedness of the solution at the point of transition from slippage to bonding, while the length of the corresponding zone in the problem on an interface crack is found from the boundedness when passing from slippage to separation.

Spence (1973) proved that there is a denumerable set of solutions when the stresses are restricted at the point of transition from bonding to slippage. Nevertheless, only the greatest sliding zone out of the possible ones leads to the solution that satisfies all additional conditions of the problem. With the help of piecewise constant interpolation of the contact stresses, Spence obtained a numerical solution of the system of Volterra's equation, to which the problem is equivalent.

Antipov and Arutyunyan (1991, 1992) constructed an analytical solution of Galin's problem in the symmetric and non-symmetric case for a half-plane and a wedge, for a stamp with straight horizontal and wedge-shaped base. In the present paper an analytical solution of Galin's problem for a periodic system of stamps is presented.

2. Formulation of the problem

Let us consider a periodic system of stamps

$$C = (-a, a) \cup (\pm 2c \mp a, \pm 2c \pm a) \cup (\pm 4c \mp a, \pm 4c \pm a) \cup \ldots \tag{2.1}$$

pressed down onto an elastic half-plane ($|x| < \infty, -\infty < y < 0$) with Young's modulus E and Poisson's ration ν under the action of vertical force P applied to the centre of each stamp (Fig. 1). The contact area S is divided into zones of adhesion

$$A = (-b, b) \cup (\pm 2c \mp b, \pm 2c \pm b) \cup (\pm 4c \mp b, \pm 4c \pm b) \cup \cdots \tag{2.2}$$

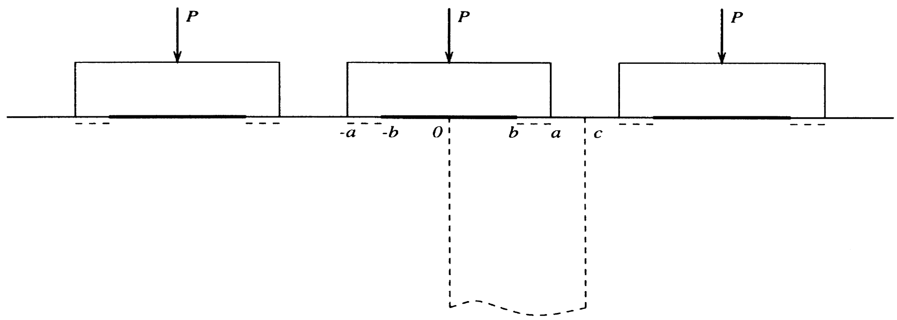

Fig. 1. Geometry of the problem: a half-plane with the periodic system of stamps.

and zones of slippage $C\backslash A$ where the conditions of dry friction apply. The quantity b has to be determined. Outside the contact zone, the half-plane boundary is free from loading.

By taking into account the periodic properties we formulate the above stated problem as the boundary value problem of the theory of two-dimensional elasticity for the half-strip $(0 < x < c, -\infty < y < 0)$

$$y = 0: \quad \sigma_y = 0, \quad \tau_{xy} = 0, \quad a < x < c; \quad \tau_{xy} + \mu\sigma_y = 0, \quad b < x < a, \tag{2.3}$$

$$y = 0: \quad u = 0, \quad 0 < x < b; \quad v = -\delta, \quad 0 < x < a, \tag{2.4}$$

$$x = 0: \quad u = 0, \quad \tau_{xy} = 0, \quad -\infty < y < 0, \tag{2.5}$$

$$x = c: \quad u = 0, \quad \tau_{xy} = 0, \quad -\infty < y < 0, \tag{2.6}$$

where δ is an additive constant and μ is the coefficient of friction. The following condition of stress equilibrium has to be satisfied

$$\int_0^a \sigma_y(x, 0)\, \mathrm{d}x = -\frac{P}{2}. \tag{2.7}$$

The shear stresses in the adhesive region are too small to create slippage: $\tau_{xy} < -\mu\sigma_y$. The normal stresses everywhere in the contact area must be negative.

Let $\sigma(x)$ and $\tau(x)$ denote the contact stresses

$$\sigma(x) = \sigma_y(x, 0), \quad \tau(x) = \tau_{xy}(x, 0). \tag{2.8}$$

The above problem can be reduced to the boundary value problem for the biharmonic function in the half-strip

$$\Delta^2 U(x, y) = 0, \quad 0 < x < c, \quad -\infty < y < 0, \tag{2.9}$$

$$\frac{\partial^2 U}{\partial x^2}(x, 0) = \sigma(x), \quad -\frac{\partial^2 U}{\partial x\, \partial y}(x, 0) = \tau(x), \quad 0 < x < c, \tag{2.10}$$

$$\operatorname{supp} \sigma(x) \subset [0, a], \quad \operatorname{supp} \tau(x) \subset [0, a], \tag{2.11}$$

$$\frac{\partial U}{\partial x}(0, y) = \frac{\partial U}{\partial x}(c, y) = 0, \quad \frac{\partial^3 U}{\partial x^3}(0, y) = \frac{\partial^3 U}{\partial x^3}(c, y) = 0, \quad -\infty < y < 0\,, \tag{2.12}$$

where $U(x, y)$ is the Airy stress function.

3. Reduction to a system of singular integral equations

Applying the finite cosine-transformation with respect to x

$$U_n(y) = \int_0^x U(x, y) \cos \beta n x \, \mathrm{d}x, \quad \beta = \frac{\pi}{c}, \tag{3.1}$$

$$U(x, y) = \frac{1}{c} U_0(y) + \frac{2}{c} \sum_{n=1}^{\infty} U_n(y) \cos \beta n x \tag{3.2}$$

to the boundary value problem (2.9)–(2.12) we arrive at the following problem for the one-dimensional equation

$$\left(\frac{\mathrm{d}^4}{\mathrm{d}y^4} - 2\beta^2 n^2 \frac{\mathrm{d}^2}{\mathrm{d}y^2} + \beta^4 n^4\right) U_n(y) = 0, \quad -\infty < y < 0\,, \tag{3.3}$$

$$U_n(0) = -\frac{\sigma_n}{\beta^2 n^2}, \quad \frac{\mathrm{d}}{\mathrm{d}y} U_n(0) = \frac{\tau_n}{\beta n}, \quad U_n(y) \to 0, \quad y \to -\infty, \tag{3.4}$$

$$\sigma_n = \int_0^a \sigma(x) \cos \beta n x \, \mathrm{d}x, \quad \tau_n = \int_0^a \tau(x) \sin \beta n x \, \mathrm{d}x, \quad n = 1, 2, \ldots. \tag{3.5}$$

The solution of this problem that vanishes as $y \to -\infty$ has the form

$$U_n(y) = \frac{(-1 + n\beta y)\, \mathrm{e}^{n\beta y}}{n^2 \beta^2} \sigma_n + \frac{y\, \mathrm{e}^{n\beta y}}{n\beta}, \quad n = 1, 2, \ldots. \tag{3.6}$$

Obviously, in the case $n = 0$ there is no non-trivial solution of eqn (3.3) under the additional condition $U_0(y) \to 0$, $y \to -\infty$. So that $U_0(y) \equiv 0$. Next, inverting the integral transformation, we use the formulae which connect the displacement with the Airy function (plane deformation)

$$\frac{1}{\nu_*} \frac{\partial u}{\partial x} = (1 - \nu) \frac{\partial^2 U}{\partial y^2} - \nu \frac{\partial^2 U}{\partial x^2}, \quad \frac{1}{\nu_*} \frac{\partial v}{\partial y} = (1 - \nu) \frac{\partial^2 U}{\partial x^2} - \nu \frac{\partial^2 U}{\partial y^2}, \quad \nu_* = \frac{1 + \nu}{E} \tag{3.7}$$

and obtain the following relationships

$$\frac{1}{\nu_*} \frac{\partial u}{\partial x}(x, y) = \frac{2}{c} \sum_{n=1}^{\infty} \cos n\beta x \, \mathrm{e}^{n\beta y} \left[(\kappa_- + n\beta y) \sigma_n + (\kappa_+ + n\beta y) \tau_n \right], \tag{3.8}$$

$$\frac{1}{\nu_*}\frac{\partial v}{\partial x}(x, y) = \frac{2}{c}\sum_{n=1}^{\infty} \sin n\beta x\, \mathrm{e}^{n\beta y}\big[(-\kappa_- + n\beta y)\tau_n + (-\kappa_+ + n\beta y)\sigma_n\big], \tag{3.9}$$

where $\kappa_\pm = (\kappa \pm 1)/2$, $\kappa = 3 - 4\nu$.

Substituting expressions (3.5) into (3.8) we change the order of the integration and summation and arrive at the representation for the derivative of the tangential displacement

$$\begin{aligned}\frac{1}{\nu_*}\frac{\partial v}{\partial x}(x, y) = {} & \frac{2}{c}\int_0^a \sigma(\xi)\sum_{n=1}^{\infty} \cos n\beta x \cos n\beta\xi\, \mathrm{e}^{n\beta y}(\kappa_- + n\beta y)\, \mathrm{d}\xi \\ & + \frac{2}{c}\int_0^a \tau(\xi)\sum_{n=1}^{\infty} \cos n\beta x \sin n\beta\xi\, \mathrm{e}^{n\beta y}(\kappa_+ + n\beta y)\, \mathrm{d}\xi.\end{aligned} \tag{3.10}$$

We summarise the series in (3.10) using the formulae (1.447 1, 2) (Gradshtein and Ryzhik, 1965) and transform expression (3.10) to the form

$$\frac{1}{\nu_*}\frac{\partial u}{\partial x} = \frac{1}{c}\int_0^a \big[S_1^+(\xi - x, y) + S_1^+(\xi + x, y)\big]\sigma(\xi)\, \mathrm{d}\xi + \frac{1}{c}\int_0^a \big[S_2^+(\xi - x, y) + S_2^+(\xi + x, y)\big]\tau(\xi)\, \mathrm{d}\xi. \tag{3.11}$$

The tangentional derivative of the normal displacement can be calculated in a similar way

$$\frac{1}{\nu_*}\frac{\partial v}{\partial x} = \frac{1}{c}\int_0^a \big[S_2^-(\xi - x, y) - S_2^-(\xi + x, y)\big]\sigma(\xi)\, \mathrm{d}\xi + \frac{1}{c}\int_0^a \big[S_1^-(\xi - x, y) - S_1^-(\xi + x, y)\big]\tau(\xi)\, \mathrm{d}\xi. \tag{3.12}$$

The following notations are assumed in the relations (3.11) and (3.12)

$$S_1^\pm(t, \xi) = \pm\frac{\kappa_-\, \mathrm{e}^{\beta y}\big(\cos\beta t - \mathrm{e}^{\beta y}\big)}{1 - 2\,\mathrm{e}^{\beta y}\cos\beta t + \mathrm{e}^{2\beta y}} + \frac{\beta y\, \mathrm{e}^{\beta y}\cos\beta t\big(1 + \mathrm{e}^{2\beta y}\big) - 2\,\mathrm{e}^{\beta y}}{\big(1 - 2\,\mathrm{e}^{\beta y}\cos\beta t + \mathrm{e}^{2\beta y}\big)^2}, \tag{3.13}$$

$$S_2^\pm(t, \xi) = \frac{\kappa_+\, \mathrm{e}^{\beta y}\sin\beta t}{1 - 2\,\mathrm{e}^{\beta y}\cos\beta t + \mathrm{e}^{2\beta y}} \pm \frac{\beta y\, \mathrm{e}^{\beta y}\sin\beta t\big(1 - \mathrm{e}^{2\beta y}\big)}{\big(1 - 2\,\mathrm{e}^{\beta y}\cos\beta t + \mathrm{e}^{2\beta y}\big)^2}. \tag{3.14}$$

In order to satisfy the boundary conditions (2.4), we study the behaviour of the functions (3.13) and (3.14) as $y \to -0$. Expansion of the functions $\mathrm{e}^{\beta y}$ and $\cos\beta t$ in the neighbourhood of the points $y = 0$ and $t = 0$, respectively and use of the approximation of the δ-function (see Korn and Korn, 1961) yield us

$$\frac{\mathrm{e}^{\beta y}\big(\cos\beta t - \mathrm{e}^{\beta y}\big)}{1 - 2\,\mathrm{e}^{\beta y}\cos\beta t + \mathrm{e}^{2\beta y}} \sim \frac{-\beta y - \dfrac{\beta^2}{2}\big(t^2 + y^2\big) + \cdots}{\beta^2 y^2 + \beta^2 t^2 + \cdots} \sim -\frac{y}{\beta\big(t^2 + y^2\big)} - \frac{1}{2} \sim \frac{\pi\delta(t)}{\beta} - \frac{1}{2}, \quad y \to -0. \tag{3.15}$$

In a similar way, one can obtain as $y \to -0$

$$\frac{\beta y\, \mathrm{e}^{\beta y}\cos\beta t\big(1 + \mathrm{e}^{2\beta y}\big) - 2\,\mathrm{e}^{\beta y}}{\big(1 - 2\,\mathrm{e}^{\beta y}\cos\beta t + \mathrm{e}^{2\beta y}\big)^2} \sim \frac{y}{\beta}\frac{y^2 - t^2}{\big(y^2 + t^2\big)^2} \sim \frac{1}{\beta}\left[\frac{y}{y^2 + t^2} - \frac{2yt^2}{\big(y^2 + t^2\big)^2}\right] \sim \frac{\pi}{\beta}\big[-\delta(t) - t\delta'(t)\big] = 0, \tag{3.16}$$

$$\frac{e^{\beta y}\sin\beta t}{1-2\,e^{\beta y}\cos\beta t+e^{2\beta y}}\sim\frac{1}{2}\cot\frac{\beta t}{2}, \tag{3.17}$$

$$\frac{\beta y\,e^{\beta y}\sin\beta t(1-e^{2\beta y})}{\left(1-2\,e^{\beta y}\cos\beta t+e^{2\beta y}\right)^2}\sim 0, \tag{3.18}$$

we arrive at the following integral representations for the derivatives of the displacements

$$\frac{1}{\nu_*}\frac{\partial u}{\partial x}(x,\,-0)=\kappa_-\sigma(x)+\frac{\kappa_+}{\pi}\int_0^a l_+(x,\,\xi)\tau(\xi)\,d\xi-\frac{\kappa_-}{c}\int_0^a\sigma(\xi)\,d\xi, \tag{3.19}$$

$$\frac{1}{\nu_*}\frac{\partial v}{\partial x}(x,\,-0)=-\kappa_-\tau(x)+\frac{\kappa_+}{\pi}\int_0^a l_-(x,\,\xi)\sigma(\xi)\,d\xi, \tag{3.20}$$

where

$$l_\pm(x,\,\xi)=\frac{\beta}{2}\left[\cot\frac{\beta}{2}(\xi-x)\pm\cot\frac{\beta}{2}(\xi+x)\right]. \tag{3.21}$$

Let us use the third condition in (2.3) of dry friction $\tau_{xy}+\mu\sigma_{xy}=0$, that is true on the segment $(b,\,a)$. We introduce the new function

$$\tau_1(x)=\tau(x)+\mu\sigma(x),\quad \operatorname{supp}\tau_1(x)\subset[0,\,b]. \tag{3.22}$$

Taking into account the equilibrium condition (2.7) we rewrite formulae (3.19) and (3.20) in terms of the functions $\sigma(x)$ and $\tau_1(x)$

$$\frac{1}{\nu_*}\frac{\partial u}{\partial x}(x,\,-0)=\kappa_-\sigma(x)+\frac{\kappa_+}{\pi}\int_0^a l_+(x,\,\xi)\left[\tau_1(\xi)-\mu\sigma(\xi)\right]d\xi+\frac{P\kappa_-}{2c}, \tag{3.23}$$

$$\frac{1}{\nu_*}\frac{\partial v}{\partial x}(x,\,-0)=-\kappa_-\tau_1(x)+\kappa_-\mu\sigma(x)+\frac{\kappa_+}{\pi}\int_0^a l_-(x,\,\xi)\sigma(\xi)\,d\xi. \tag{3.24}$$

Satisfaction of the boundary conditions

$$\frac{\partial u}{\partial x}(x,\,-0)=0,\quad 0<x<b;\quad \frac{\partial v}{\partial x}(x,\,-0)=0,\quad 0<x<a \tag{3.25}$$

and use of the property (3.22) yield the following system of two integral equations on different segments with Hilbert's kernels

$$\kappa_-\sigma(x)-\frac{\mu\kappa_+}{\pi}\int_0^a l_+(x,\,\xi)\sigma(\xi)\,d\xi+\frac{\kappa_+}{\pi}\int_0^b l_+(x,\,\xi)\tau_1(\xi)\,d\xi=-\frac{P\kappa_-}{2c},\quad 0<x<b, \tag{3.26}$$

$$\kappa_-\mu\sigma(x)+\frac{\kappa_+}{\pi}\int_0^a l_-(x,\,\xi)\sigma(\xi)\,d\xi-\kappa_-\tau_1(\xi)=0,\quad 0<x<a. \tag{3.27}$$

4. Analysis of the system of integral equations

The contact problem in the half-strip (2.3)–(2.6) has been reduced to the system of integral equations (3.26) and (3.27). Let us introduce the class of solutions. A function $\phi(\xi)$, which satisfies everywhere on $(0, l)$, except possibly its ends, the Hölder condition, has at the points $\xi = 0$ and $\xi = l$ an integrable singularity

$$\phi(\xi) = A_0\xi^{-\alpha_0}, \quad \xi \to +0, \quad \phi(\xi) = A_1(l-\xi)^{-\alpha_1}, \quad \xi \to l-0,$$

$$A_j = \text{const}, \quad \Re\alpha_j \in (0, 1), \quad j = 0, 1, \tag{4.1}$$

is said to belong to the class $h^{\alpha_0,\alpha_1}(0, l)$. The functions $\sigma(\xi)$, $\tau_1(\xi)$ are sought in the following class

$$\sigma(\xi) \in h^{\alpha_0,\alpha_1}(0, b) \cup h^{\alpha_1,\alpha_2}(b, a), \quad \tau_1(\xi) \in h^{\alpha_0,\alpha_1}(0, b). \tag{4.2}$$

In other words, we admit integrable singularities for unknown functions at the points 0, b and a. In order to define the quantities α_0, α_1 and α_2 a priori, we split the Hilbert kernel into the Cauchy kernel and a regular function

$$\cot x = \frac{1}{x} + B(x), \tag{4.3}$$

where

$$B(x) = -\frac{x}{3} - \frac{x^3}{45} - \cdots \frac{2^{2n}B_n x^{2n-1}}{(2n)!} - \cdots, x^2 < \pi^2 \tag{4.4}$$

and B_n are Bernoulli's numbers. Next, we use the Muskhelishvili formulae (Muskhelishvili, 1953; Gakhov,1966) that define the behaviour of the Cauchy type integral at the points at which the density has a power singularity

$$\frac{1}{\pi}\int_{c_1}^{c_2} \frac{\phi^*(\xi)\,d\xi}{(\xi-c_0)^\delta(\xi-x)} = \left[\frac{e^{\delta\pi i}}{\sin\delta\pi}\phi^*(c_0+0) - \cot\delta\pi\phi^*(c_0-0)\right](x-c_0)^{-\delta} + Y^{(1)}(x), \quad x \to c_0 - 0, \tag{4.5}$$

$$\frac{1}{\pi}\int_{c_1}^{c_2} \frac{\phi^*(\xi)\,d\xi}{(\xi-c_0)^\delta(\xi-x)} = \left[\cot\delta\pi\phi^*(c_0+0) - \frac{e^{-\delta\pi i}}{\sin\delta\pi}\phi^*(c_0-0)\right](x-c_0)^{-\delta} + Y^{(2)}(x), \quad x \to c_0 + 0, \tag{4.6}$$

where $\phi^*(\xi)$ is a function which satisfies the Hölder condition on the closed segment $[c_1, c_2]$, $c_1 \leqslant c_0 \leqslant c_2$, $\delta = \delta_1 + i\delta_2$, $0 \leqslant \delta_1 < 1$; $Y^{(1)}(x)$, $Y^{(2)}(x)$ are functions analytic in the vicinity of the point c_0. The values of the function $(x-c_0)^{-\gamma}$ are specified by

$$(x-c_0)^{-\delta} = |x-c_0|^{-\delta}, \quad x > c_0, \quad (x-c_0)^{-\delta} = e^{-\delta\pi i}|x-c_0|^{-\delta}, \quad x < c_0. \tag{4.7}$$

Let us assume that

$$\sigma(\xi) \sim A_0\xi^{-\alpha_0}, \quad \tau_1(\xi) \sim B_0\xi^{-\alpha_0}, \quad \xi \to +0, \tag{4.8}$$

$$\sigma(\xi) \sim A_1^{\pm}(\xi-b)^{-\alpha_1}, \quad \xi \to b \pm 0; \quad \tau_1(\xi) \sim B_1^{\pm}(\xi-b)^{-\alpha_1}, \quad \xi \to b-0, \tag{4.9}$$

$$\sigma(\xi) \sim A_2(\xi - a)^{-\alpha_2}, \quad \xi \to a - 0. \tag{4.10}$$

Using formula (4.5) in the case $c_1 = 0$, $c_0 = c_2 = a$ from eqn (3.27) we obtain as $x \to a - 0$

$$(\mu\kappa_- - \kappa_+ \cot \alpha_2\pi)A_2(x - a)^{-\alpha_2} = \Omega_0(x), \quad x \to a, \tag{4.11}$$

where $\Omega_0(x)$ is an analytic function in the neighbourhood of the point a. Eqn (4.11) yields immediately

$$\mu\kappa_- - \kappa_+ \cot \alpha_2\pi = 0. \tag{4.12}$$

Taking into account that $\Re\alpha_2 \in (0, 1)$ we find

$$\alpha_2 = \frac{1}{\pi} \tan^{-1} \frac{1}{\mu\gamma}, \quad \gamma = \frac{\kappa_-}{\kappa_+}. \tag{4.13}$$

Now we consider the vicinity of the point b. At first, we apply formulae (4.5) and (4.6) to the analysis of the behaviour of the left-hand-side of eqns (3.26) and (3.27) as $x \to b - 0$

$$\left[\kappa_- A_1^- - \mu\kappa_+\left(\frac{e^{\alpha_1\pi i}}{\sin \alpha_1\pi} A_1^+ - \cot \alpha_1\pi A_1^-\right) - \kappa_+ \cot \alpha_1\pi B_1\right](x - b)^{-\alpha_1} = \Omega_1(x), \tag{4.14}$$

$$\left[\kappa_-\mu A_1^- + \kappa_+\left(\frac{e^{\alpha_1\pi i}}{\sin \alpha_1\pi} A_1^+ - \cot \alpha_1\pi A_1^-\right) - \kappa_- B_1\right](x - b)^{-\alpha_1} = \Omega_2(x). \tag{4.15}$$

On the other hand, eqn (3.27) yields as $x \to b + 0$

$$\left[(\kappa_-\mu + \kappa_+ \cot \alpha_1\pi)A_1^+ - \frac{\kappa_+ e^{-\alpha_1\pi i}}{\sin \alpha_1\pi} A_1^-\right](x - b)^{-\alpha_1} = \Omega_3(x). \tag{4.16}$$

The functions $\Omega_m(x)\,(m = 1, 2, 3)$ are analytic in the vicinity of the point $x = b$. Consequently, the expressions in the square brackets should be equal to 0. A non-trivial solution of this homogeneous algebraic system with respect to A_1,A_2 and A_3 exists if and only if the determinant

$$\Delta(\alpha_1) = -\kappa(\kappa_-\mu + \kappa_+ \cot \alpha_1\pi) \tag{4.17}$$

of the system is equal to 0. The equation $\Delta(\alpha_1) = 0$ in the strip $0 < \Re\alpha_1 < 1$ has a single root

$$\alpha_1 = 1 - \frac{1}{\pi} \tan^{-1} \frac{1}{\mu\gamma}. \tag{4.18}$$

Let us consider the point $x = 0$. Formulae (4.5) and (4.6) and eqns (3.26) and (3.27) yield the following system of two equations

$$\left(\kappa_- - \mu\kappa_+ \cot \frac{\alpha_0\pi}{2}\right)A_0 + \kappa_+ \cot \frac{\alpha_0\pi}{2} B_0 = 0, \tag{4.19}$$

$$\left(\kappa_-\mu - \kappa_+ \tan \frac{\alpha_0\pi}{2}\right)A_0 - \kappa_- B_0 = 0. \tag{4.20}$$

Since the determinant of this system is equal to κ and $\kappa \neq 0$ we see that both functions $\sigma(\xi)$ and $\tau_1(\xi)$ as well as their derivatives cannot have power singularities at the point 0. On the other hand, the physical sense of the functions $\sigma(\xi)$ and $\tau_1(\xi)$ leads to boundedness of these functions at the point 0. The same result can be obtained if we assume $\alpha_0 = 0$ in formulae (4.8)–(4.10) and apply the analogue of relations (4.5) and (4.6) for the case $\delta = 0$ (see Gakhov, 1966) to eqns (3.26) and (3.27).

Thus, the functions $\sigma(\xi)$ and $\tau_1(\xi)$ should be determined in the class

$$\sigma(\xi) \in h^{0,1-\alpha}(0, b) \cup h^{1-\alpha,\alpha}(b, a), \quad \tau_1(\xi) \in h^{0,1-\alpha}(0, b), \tag{4.21}$$

$$\alpha = \frac{1}{\pi} \tan^{-1} \frac{1}{\mu\gamma} \in (0, 1). \tag{4.22}$$

Additionally, the function $\sigma(\xi)$ satisfies the complementary condition

$$\int_0^a \sigma(\xi)\, \mathrm{d}\xi = -\frac{P}{2}. \tag{4.23}$$

The solution of system (3.26) and (3.27) can be constructed numerically if we split the Hilbert kernels into two parts (4.3) and (4.4) where for large values of x we can assume $B(x) = \cot x - x^{-1}$. Then, the numerical technique that was used by Spence (see Spence, 1972) for the corresponding homogeneous characteristic system with Cauchy kernels, can be applied. We propose an analytical approach.

5. Solution of the system of integral equations

In order to reduce system (3.26) and (3.27) to a matrix Wiener–Hopf problem, we should transform the equations with Hilbert kernels to the Mellin convolution type equations. Let us introduce the new variables and parameters

$$\tau = \tan \frac{\beta\xi}{2}, \quad t = \tan \frac{\beta x}{2},$$

$$a_0 = \tan \frac{a\beta}{2}, \quad b_0 = \tan \frac{b\beta}{2} \tag{5.1}$$

and the new unknown functions $\chi_1(\tau)$, $\chi_2(\tau)$ connected with the old ones through the relationships

$$\sigma(\xi) = \frac{P}{2c \cos^2(\beta\xi/2)} \chi_1\left(\tan \frac{\beta\xi}{2}\right), \quad \tau_1(\xi) = \frac{P}{2c \cos^2(\beta\xi/2)} \chi_2\left(\tan \frac{\beta\xi}{2}\right). \tag{5.2}$$

Then taking into account the identities

$$\cot \frac{\beta}{2}(\xi \pm x) = \mp t + \frac{t^2 + 1}{\tau \pm t}, \quad \cos^2 \frac{\beta x}{2} = \frac{1}{t^2 + 1} \tag{5.3}$$

as well as the condition

$$\int_0^{a_0} \sigma\left(\frac{2}{\beta} \tan^{-1} \tau\right) \frac{\mathrm{d}\tau}{\tau^2 + 1} = -\frac{P\beta}{4}, \tag{5.4}$$

that follows from (4.23), we reduce system (3.26) and (3.27) to the system of integral equations with Cauchy kernels

$$\gamma\chi_1(t) - \frac{\mu}{\pi}\int_0^{a_0} \frac{2\tau\chi_1(\tau)}{\tau^2 - t^2}\,\mathrm{d}\tau + \frac{1}{\pi}\int_0^{b_0} \frac{2\tau\chi_2(\tau)}{\tau^2 - t^2}\,\mathrm{d}\tau = -\frac{\gamma}{t^2+1}, \quad 0 < t < b_0, \tag{5.5}$$

$$\gamma\mu\chi_1(t) - \gamma\chi_2(t) + \frac{1}{\pi}\int_0^{a_0} \frac{2t\chi_1(\tau)}{\tau^2 - t^2}\,\mathrm{d}\tau = \frac{t}{t^2+1}, \quad 0 < t < a_0, \tag{5.6}$$

where γ was determined in (4.13) and the unknown functions are expressed through the contact stresses as follows

$$\chi_1(\tau) = \frac{2c}{P(\tau^2+1)}\sigma\left(\frac{2}{\beta}\tan^{-1}\tau\right), \tag{5.7}$$

$$\chi_2(\tau) = \frac{2c}{P(\tau^2+1)}\tau_1\left(\frac{2}{\beta}\tan^{-1}\tau\right). \tag{5.8}$$

According to the analysis in the previous section, the solution of system (5.5) and (5.6) must be sought in the class

$$\chi_1(\tau) \in h^{0,1-\alpha}(0, b_0) \cup h^{1-\alpha,\alpha}(b_0, a_0), \quad \chi_2(\tau) \in h^{0,1-\alpha}(0, b_0), \tag{5.9}$$

whereas the function $\chi_1(\tau)$ satisfies the complementary condition

$$\int_0^{a_0} \chi_1(\tau)\,\mathrm{d}\tau = -\frac{\pi}{2}. \tag{5.10}$$

First, extend the definition of system (5.5) and (5.6) over the entire positive half-axis using the functions $\chi_{1+}(t)$ and $\chi_{2+}(t)$ so that $\operatorname{supp}\chi_{1+} \subset [b_0, \infty)$, $\operatorname{supp}\chi_{2+} \subset [a_0, \infty)$ and

$$\gamma\chi_{1-}(t) - \frac{2\mu}{\pi}\int_0^{\infty} \frac{\chi_{1-}(\tau)}{1-\left(\dfrac{t}{\tau}\right)^2}\frac{\mathrm{d}\tau}{\tau} + \frac{2}{\pi}\int_0^{\infty} \frac{\chi_{2-}(\tau)}{1-\left(\dfrac{t}{\tau}\right)^2}\frac{\mathrm{d}\tau}{\tau} = -\frac{\gamma}{t^2+1} + \chi_{1+}(t), \quad 0 < t < \infty, \tag{5.11}$$

$$\mu\gamma\chi_{1-}(t) - \gamma\chi_{2-}(t) + \frac{2}{\pi}\int_0^{\infty} \frac{\dfrac{t}{\tau}\chi_{1-}(\tau)}{1-\left(\dfrac{t}{\tau}\right)^2}\frac{\mathrm{d}\tau}{\tau} = \frac{t}{t^2+1} + \chi_{2+}(t), \quad 0 < t < \infty, \tag{5.12}$$

where

$$\chi_{1-}(\tau) = \begin{cases} \chi_1(\tau), & 0 < \tau < a_0 \\ 0 & \tau > a_0 \end{cases}, \quad \chi_{2-}(\tau) = \begin{cases} \chi_2(\tau), & 0 < \tau < b_0 \\ 0 & \tau > b_0 \end{cases}. \tag{5.13}$$

Next, we introduce the Mellin transforms

$$\Phi_1^-(s) = \int_0^1 \chi_{1-}(a_0\tau)\tau^s\,\mathrm{d}t, \quad \Phi_2^-(s) = \int_0^1 \chi_{2-}(b_0\tau)\tau^s\,\mathrm{d}t, \tag{5.14}$$

$$\Phi_1^+(s) = \int_1^\infty \chi_{1+}(b_0\tau)\tau^s\,dt, \quad \Phi_2^+(s) = \int_1^\infty \chi_{2+}(a_0\tau)\tau^s\,dt. \tag{5.15}$$

According to the behaviour (5.9) of the functions $\chi_j(\tau)$ at the point 0 and the asymptotics of the functions $\chi_{j+}(\tau)$ at infinity that follow from eqns (5.11) and (5.12), the unknown functions $\Phi_j^-(s)$ are analytic in the right half-plane $\Re(s) > -1$ and $\Phi_j^+(s)$ ($j = 1, 2$) are unknown analytic functions in the left half-plane $\Re(s) < 2 - j$. Taking into account the values of integrals (Gradshtein and Ryzhik, 1965)

$$\int_0^\infty \frac{x^s}{1-x^2}\,dx = -\frac{\pi}{2}\tan\frac{\pi s}{2}, \quad \int_0^\infty \frac{x^s}{1+x^2}\,dx = \frac{\pi}{2\cos\dfrac{\pi s}{2}}, \quad -1 < \Re(s) < 1, \tag{5.16}$$

we use the convolution theorem (see the Appendix) and arrive at the 2×2 matrix Wiener–Hopf problem. It is required to find two vectors: $\mathbf{\Phi}^+(s)$, analytic in the domain D^+: $\Re(s) < \gamma_0(-1 < \gamma_0 < 0)$, and $\mathbf{\Phi}^-(s)$, analytic in the domain D^-: $\Re(s) > \gamma_0$, which satisfy on the contour Γ: $\Re(s) = \gamma_0$ the following boundary condition

$$\mathbf{\Phi}^+(s) = \mathbf{G}(s)\mathbf{\Phi}^-(s) + \mathbf{f}(s), \quad s \in \Gamma, \tag{5.17}$$

$$\mathbf{G}(s) = \begin{pmatrix} \lambda^{-s-1}K_1(s) & -\tan\dfrac{\pi s}{2} \\ K_0(s) & -\gamma\lambda^{s+1} \end{pmatrix}, \quad \mathbf{\Phi}^\pm(s) = \begin{pmatrix} \mathbf{\Phi}_1^\pm(s) \\ \mathbf{\Phi}_2^\pm(s) \end{pmatrix}, \tag{5.18}$$

$$K_1(s) = \gamma + \mu\tan\frac{\pi s}{2}, \quad K_0(s) = \mu\gamma + \cot\frac{\pi s}{2}, \tag{5.19}$$

$$\mathbf{f}(s) = \begin{pmatrix} \gamma b_0^{-s-1}F(s) \\ -a_0^{-s-1}F(s+1) \end{pmatrix}, \quad F(s) = \frac{\pi}{2\cos\dfrac{\pi s}{2}}, \tag{5.20}$$

$$\lambda = \frac{b_0}{a_0}, \quad 0 < \lambda < 1. \tag{5.21}$$

Derive the function $\Phi_1^-(s)$ from the second equation in system (5.17)

$$\Phi_1^-(s) = \frac{\Phi_2^+(s) + a_0^{-s-1}F(s+1) + \gamma\lambda^{s+1}\Phi_2^-(s)}{K_0(s)} \tag{5.22}$$

and substitute it into the first one. The matrix Wiener–Hopf problem can be rewritten in the different form

$$\lambda^{s+1}\Phi_1^+(s) - \gamma a_0^{-s-1}F(s) = \frac{K_1(s)}{K_0(s)}\left[\Phi_2^+(s) + a_0^{-s-1}F(s+1)\right] - \frac{\gamma_*\lambda^{s+1}}{K_0(s)}\Phi_2^-(s), \tag{5.23}$$

$$\Phi_2^+(s) + a_0^{-s-1}F(s+1) = K_0(s)\Phi_1^-(s) - \gamma\lambda^{s+1}\Phi_2^-(s), \quad s \in \Gamma, \tag{5.24}$$

where $\gamma_* = 1 - \gamma^2$. In order to factorize the function $K_0(s)$, we present it as follows

$$K_0(s) = \frac{\sin \pi\left(\frac{s}{2}+\alpha\right)}{\sin \frac{\pi s}{2} \sin \pi\alpha}, \tag{5.25}$$

where the parameter α is the same as that in (4.22). Further, if we use the formula

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}, \tag{5.26}$$

where $\Gamma(x)$ is the Euler integral of the second kind (the Gamma function) then we arrive immediately at the following splitting of the function $K_0(s)$

$$K_0(s) = K_0^+(s)K_0^-(s), \quad s \in \Gamma, \tag{5.27}$$

$$K_0^+(s) = -\frac{\kappa_0\Gamma\left(-\frac{s}{2}\right)}{\Gamma\left(1-\alpha-\frac{s}{2}\right)}, \quad K_0^-(s) = \frac{\Gamma\left(1+\frac{s}{2}\right)}{\Gamma\left(\alpha+\frac{s}{2}\right)}, \quad \kappa_0 = \frac{1}{\sin \pi\alpha}. \tag{5.28}$$

Next, we substitute the splitting (5.27) into eqns (5.23) and (5.24) and multiply both parts of these equations by the function $\lambda^{-s-1}K_0^+(s)$ and $[K_0^+(s)]^{-1}$, respectively. Thus, we transform the eqns (5.23) and (5.24) into the form

$$\frac{-\gamma_*\Phi_2^-(s)}{K_0^-(s)} = K_0^+(s)\Phi_1^+(s) - \frac{\pi\gamma b_0^{-s-1}K_0^+(s)}{2\cos\frac{\pi s}{2}} - \frac{K_1(s)\lambda^{-s-1}}{K_0^-(s)}\left[\Phi_2^+(s) - \frac{\pi a_0^{-s-1}}{2\sin\frac{\pi s}{2}}\right], \tag{5.29}$$

$$\frac{\Phi_2^+(s)}{K_0^+(s)} - \frac{\pi a_0^{-s-1}}{2\sin\frac{\pi s}{2}K_0^+(s)} = K_0^-(s)\Phi_1^-(s) - \frac{\gamma\lambda^{s+1}\Phi_2^-(s)}{K_0^+(s)}, \quad s \in \Gamma. \tag{5.30}$$

Let us find sectionally-analytic functions $\Omega_j(s)$ $(j = 1, 2)$ vanishing at infinity and undergoing in passing through the contour Γ jumps

$$\Omega_1^+(s) - \Omega_1^-(s) = \frac{\pi b_0^{-s-1}}{2}\left[\frac{K_1(s)}{K_0^-(s)\sin\frac{\pi s}{2}} - \frac{\lambda K_0^+(s)}{\cos\frac{\pi s}{2}}\right], \quad s \in \Gamma, \tag{5.31}$$

$$\Omega_2^+(s) - \Omega_2^-(s) = \frac{-\pi a_0^{-s-1}}{2K_0^+(s)\sin\frac{\pi s}{2}}, \quad s \in \Gamma. \tag{5.32}$$

In the case $b_0 > 1$ for the function $\Omega_1^+(s)$ the expansion in simple fractions takes place

$$\Omega_1^+(s) = \sum_{j=1}^{\infty}\frac{C_j^+}{s-2j+1}, \quad s \in D^+, \quad C_j^+ = \frac{\mu\gamma_*(-1)^j b_0^{-2j}\Gamma\left(\alpha+j-\frac{1}{2}\right)}{\Gamma\left(j+\frac{1}{2}\right)}. \tag{5.33}$$

Obviously, the function $\Omega_1^-(s)$ admits the analytical continuation in the right half-plane D^- that is defined by the relationship

$$\Omega_1^-(s) = \Omega_1^+(s) - \frac{\pi\mu\gamma_* b_0^{-s-1}}{2K_0^-(s)\cos\dfrac{\pi s}{2}}, \quad s \in D^-, \tag{5.34}$$

which follows directly from (5.31). In the case $b_0 < 1$ the function $\Omega_1^-(s)$ has the form

$$\Omega_1^-(s) = -\sum_{j=1}^{\infty}\left(\frac{C_j^-}{s+2j-1} + \frac{D_j^-}{s+2j+2\alpha-2}\right), \quad s \in D^-, \tag{5.35}$$

$$C_j^- = \frac{\kappa_0\gamma_* b_0^{2j-2}\Gamma\left(j-\frac{1}{2}\right)}{(-1)^j\gamma\Gamma\left(j-\alpha+\frac{1}{2}\right)}, \quad D_j^- = \frac{\gamma_* b_0^{2j+2\alpha-3}\Gamma(j-1+\alpha)}{(-1)^{j-1}\gamma\Gamma(j)}, \tag{5.36}$$

and the analytical continuation for the function $\Omega_1^+(s)$ follows directly from (5.34). Similarly, for the functions $\Omega_2^\pm(s)$ for $a_0 > 1$ we have

$$\Omega_2^+(s) = \sum_{j=1}^{\infty}\frac{E_j^+}{s+2\alpha-2j}, \quad s \in D^+, \quad E_j^+ = \frac{(-1)^{j-1}a_0^{-2j+2\alpha-1}\Gamma(j+1-\alpha)}{\kappa_0\Gamma(j)}. \tag{5.37}$$

If $a_0 < 1$ then

$$\Omega_2^-(s) = -\sum_{j=1}^{\infty}\frac{E_j^-}{s+2j}, \quad s \in D^-, \quad E_j^- = \frac{(-1)^j a_0^{2j-1}\Gamma(j+1-\alpha)}{\kappa_0\Gamma(j)}. \tag{5.38}$$

The function $\Omega_2^-(s)$ for $a_0 > 1$ and the function $\Omega_2^+(s)$ for $a_0 < 1$ are defined by analytical continuation from relationship (5.32). The corresponding representations can also be written for the functions

$$-\frac{K_1(s)\lambda^{-s-1}}{K_0^-(s)}\Phi_2^+(s), \quad -\frac{\gamma\lambda^{s+1}}{K_0^+(s)}\Phi_2^-(s). \tag{5.39}$$

Although these expressions contain the unknown functions $\Phi_2^+(s), \Phi_2^-(s)$ nevertheless, the poles of functions (5.39) in D^+ and D^-, respectively, are known. In contrast with (5.31) and (5.32), the coefficients of expansions (residues) in the presentations for functions (5.39) cannot be written down explicitly. Let us introduce the following functions

$$\Psi_0^-(s) = \sum_{j=1}^{\infty}\frac{A_j^-}{s+2\alpha+2j-2}, \quad \Psi_1^-(s) = \sum_{j=1}^{\infty}\frac{B_j}{s+2j-1}, \tag{5.40}$$

$$\Psi_0^+(s) = \sum_{j=1}^{\infty}\frac{A_j^+}{s+2\alpha-2j}, \tag{5.41}$$

where

$$A_j^+ = -\lim_{s\to-2\alpha+2j}\frac{(s+2\alpha-2j)\gamma\lambda^{s+1}\Phi_2^-(s)}{K_0^+(s)}, \tag{5.42}$$

$$B_j = -\lim_{s \to 1-2j} \frac{(s-1+2j)K_1(s)\Phi_2^+(s)}{\lambda^{s+1}K_0^-(s)}, \tag{5.43}$$

$$A_j^- = -\lim_{s \to -2\alpha+2-2j} \frac{(s+2\alpha-2+2j)K_1(s)\Phi_2^+(s)}{\lambda^{s+1}K_0^-(s)}. \tag{5.44}$$

Taking into account the class of solutions (5.9), according to the Muskhelishvili formulae (4.5) and (4.6) and eqns (5.11) and (5.12) we establish the behaviour of the functions $\chi_{j-}(\tau)$, $\chi_{j+}(\tau)$ $(j=1,2)$ as $\tau \to 1 \pm 0$

$$\chi_{1-}(\tau) = O\left(\{a_0-\tau\}^{-\alpha}\right), \quad \tau \to a_0 - 0, \quad \chi_{2-}(\tau) = O\left(\left\{b_0-\tau\right\}^{-1+\alpha}\right), \quad \tau \to b_0 - 0, \tag{5.45}$$

$$\chi_{1+}(\tau) = O\left(\left\{\tau_0-b_0\right\}^{-1+\alpha}\right), \quad \tau \to b_0 + 0, \quad \chi_{2+}(\tau) = O\left(\{\tau-a_0\}^{-\alpha}\right), \quad \tau \to a_0 + 0. \tag{5.46}$$

Taking into account formulae (5.45) and (5.46) we use Abelian-type theorems (formulations of these theorems are recorded in the Appendix) and arrive at the asymptotics of the unknown functions at infinity

$$\Phi_1^-(s) = O(s^{\alpha-1}), \quad \Phi_2^-(s) = O(s^{-\alpha}), \quad s \to \infty, \quad s \in D^-, \tag{5.47}$$

$$\Phi_1^+(s) = O(s^{-\alpha}), \quad \Phi_2^+(s) = O(s^{\alpha-1}), \quad s \to \infty, \quad s \in D^+. \tag{5.48}$$

Now we may estimate the behaviour of the coefficients $A_j^{\pm}$, B_j as $j \to \infty$

$$A_j^+ = O\left(\lambda^{2j} j^{1-2\alpha}\right), \quad A_j^- = O\left(\lambda^{2j} j^{-2+2\alpha}\right), \quad B_j = O\left(\lambda^{2j} j^{-2+2\alpha}\right), \quad j \to \infty. \tag{5.49}$$

Therefore, series (5.40) and (5.41) converge uniformly in the corresponding regions

$$D_0^- = \mathbf{C} \backslash \cup \left\{s \in \mathbf{C}: |s+2\alpha+2j-2| < \varepsilon\right\},$$

$$D_1^- = \mathbf{C} \backslash \cup \left\{s \in \mathbf{C}: |s+2j-1| < \varepsilon\right\},$$

$$D_0^+ = \mathbf{C} \backslash \cup \left\{s \in \mathbf{C}: |s+2\alpha-2j| < \varepsilon\right\}, \quad j = 1, 2, \ldots$$

where $\mathbf{C}$ is a complex plane, ε is a positive number as small as desired. Thus, the functions $\Psi_0^+(s)$ is analytic in D_0^+ and the functions $\Psi_0^-(s), \Psi_1^-(s)$ are analytic in D_0^-, D_1^-, respectively. We note that $D_0^{\pm} \supset D^{\pm}$, $D_1^- \supset D^-$.

Subtracting from the left- and right-hand-sides of eqn (5.29) the sum $\Psi_0^-(s) + \Psi_1^-(s)$ and of eqn (5.30) the function $\Psi_0^+(s)$, we remove the poles. Now we may apply the theorem on analytical continuation. Taking into account the behaviour (5.47) and (5.48) at infinity of the functions $\Phi_j^{\pm}$ $(j=1,2)$ as well as the asymptotic equalities as $s \to \infty$ (see, e.g., Olver, 1974)

$$K_0^+(s) \sim -\kappa_0 \left(-\frac{s}{2}\right)^{\alpha-1}, \quad s \in D^+, \quad K_0^-(s) \sim \left(\frac{s}{2}\right)^{1-\alpha}, \quad s \in D^- \tag{5.50}$$

according to Liouville's theorem (see the Appendix) we arrive at the solution of Wiener–Hopf matrix problem (5.17)

$$\Phi_1^+(s) = -\frac{\Sigma_{1+}(s)}{K_0^+(s)} + \frac{\lambda^{-s-1}K_1(s)\Sigma_{2+}(s)}{K_0^-(s)}, \quad \Phi_2^+(s) = K_0^+(s)\Sigma_{2+}(s), \tag{5.51}$$

$$\Phi_1^-(s) = \frac{\Sigma_{2-}(s)}{K_0^-(s)} + \frac{\gamma\lambda^{s+1}\Sigma_{1-}(s)}{\gamma_* K_0^+(s)}, \quad \Phi_2^-(s) = \frac{1}{\gamma_*}K_0^-(s)\Sigma_{1-}(s), \tag{5.52}$$

$$\Sigma_{1\pm}(s) = \Omega_1^\pm(s) - \Psi_0^-(s) - \Psi_1^-(s), \quad \Sigma_{2\pm}(s) = C - \Omega_2^\pm(s) + \Psi_0^+(s), \tag{5.53}$$

with C being an arbitrary constant. For the functions $\Phi_j^\pm(s)$ $(j = 1, 2)$ to be analytic in the half-planes $D^\pm$ it is necessary and sufficient that conditions (5.42)–(5.44) be satisfied. Substituting formulae (5.51) and (5.52) into conditions (5.42)–(5.44) we obtain the infinite linear algebraic system of equations with respect to the coefficients A_n^+, A_n^-, B_n

$$A_n^- = \lambda^{2n-3+2\alpha}\Delta_{0n}^+\left(C - \Omega_2^+(2-2\alpha-2n) - \sum_{m=1}^{\infty}\frac{A_m^+}{2n+2m-2}\right), \tag{5.54}$$

$$B_n = \lambda^{2n-2}\Delta_{1n}^+\left(C - \Omega_2^+(1-2n) - \sum_{m=1}^{\infty}\frac{A_m^+}{2n+2m-2\alpha-1}\right), \tag{5.55}$$

$$A_n^+ = \lambda^{2n+1-2\alpha}\Delta_n^-\left(-\Omega_1^-(2n-2\alpha) + \sum_{m=1}^{\infty}\frac{A_m^-}{2n+2m-2} + \sum_{m=1}^{\infty}\frac{B_m}{2n+2m-2\alpha-1}\right),$$

$$n = 1, 2, \ldots, \tag{5.56}$$

where

$$\Delta_{0n}^+ = -\frac{2\gamma_*\Gamma^2(n-1+\alpha)}{\pi\gamma\Gamma^2(n)}, \quad \Delta_{1n}^+ = \frac{2\kappa_0^2\Gamma^2\left(n-\frac{1}{2}\right)}{\pi\gamma\Gamma^2\left(n-\alpha+\frac{1}{2}\right)}, \tag{5.57}$$

$$\Delta_n^- = -\frac{2\gamma\Gamma^2(n+1-\alpha)}{\pi\gamma_*\kappa_0^2\Gamma(n)}. \tag{5.58}$$

In order to separate the problems on definition of the coefficients $A_n^\pm$ and B_n on the one hand and the constant C on the other hand, we present the coefficients $A_n^\pm$, B_n as follows

$$A_n^\pm = CA_{n0}^\pm + A_{n1}^\pm, \quad B_n = CB_{n0} + B_{n1}. \tag{5.59}$$

The substitution of (5.59) into eqns (5.54)–(5.56) yields the system of equations with respect to the new coefficients $A_{nj}^\pm$, A_{nj}^-, B_{nj}

$$A_{nj}^- = \lambda^{2n-3+2\alpha}\Delta_{0n}^+\left(f_{nj}^- - \sum_{m=1}^{\infty}\frac{A_{mj}^+}{2n+2m-2}\right), \tag{5.60}$$

$$B_{nj} = \lambda^{2n-2}\Delta_{1n}^{+}\left(f_{nj} - \sum_{m=1}^{\infty}\frac{A_{mj}^{+}}{2n+2m-2\alpha-1}\right), \tag{5.61}$$

$$A_{nj}^{+} = \lambda^{2n+1-2\alpha}\Delta_{n}^{-}\left(f_{nj}^{+} + \sum_{m=1}^{\infty}\frac{A_{mj}^{-}}{2n+2m-2} + \sum_{m=1}^{\infty}\frac{B_{mj}}{2n+2m-2\alpha-1}\right),$$

$$n = 1, 2, \ldots; \quad j = 0, 1, \tag{5.62}$$

$$f_{n0}^{-} = 1, \quad f_{n0} = 1, \quad f_{n0}^{+} = 0, \tag{5.63}$$

$$f_{n1}^{-} = -\Omega_2^{+}(2-2\alpha-2n), \quad f_{n1} = -\Omega_2^{+}(1-2n), \quad f_{n1}^{+} = -\Omega_1^{-}(2n-2\alpha). \tag{5.64}$$

This system is a normal-type infinite system (of the Poincare–Koch-type) and does not involve the unknown constant C. Due to the particular structure of system (5.60)–(5.62), there are two effective ways for its solution. The first one is the reduction method (see Kantorovich and Krylov, 1964)

$$A_{nj}^{-(N)} = \lambda^{2n-3+2\alpha}\Delta_{0n}^{+}\left(f_{nj}^{-} - \sum_{m=1}^{N}\frac{A_{mj}^{+(N)}}{2n+2m-2}\right), \tag{5.65}$$

$$B_{nj}^{(N)} = \lambda^{2n-2}\Delta_{1n}^{+}\left(f_{nj} - \sum_{m=1}^{N}\frac{A_{mj}^{+(N)}}{2n+2m-2\alpha-1}\right), \tag{5.66}$$

$$A_{nj}^{+(N)} = \lambda^{2n+1-2\alpha}\Delta_{n}^{-}\left[f_{nj}^{+} + \sum_{m=1}^{N}\left(\frac{A_{mj}^{-(N)}}{2n+2m-2} + \frac{B_{mj}^{(N)}}{2n+2m-2\alpha-1}\right)\right],$$

$$n = 1, 2, \ldots, N; \quad j = 0, 1. \tag{5.67}$$

The exponential convergence of an approximate solution to the exact one is assured by the exponential decay of the non-diagonal elements of the matrix of system (5.65)–(5.67). In a number of cases it is convenient to convert system (5.60)–(5.62) through recurrence relations. We represent the coefficients $A_{nj}^{\pm}$, B_{nj} as expansions in the parameter λ

$$A_{nj}^{-} = \lambda^{2n-3+2\alpha}\sum_{k=1}^{\infty}a_{nkj}^{-}\lambda^{2k-2}, \quad B_{nj} = \lambda^{2n-2}\sum_{k=1}^{\infty}b_{nkj}\lambda^{2k-2}, \tag{5.68}$$

$$A_{nj}^{+} = \lambda^{2n}\sum_{k=1}^{\infty}a_{nkj}^{+}\lambda^{2k-2}. \tag{5.69}$$

Substituting the last relationships into system (5.60)–(5.62) we get the simple recurrence relations

$$a_{n1j}^{-} = \Delta_{0n}^{+}f_{nj}^{-}, \quad b_{n1j} = \Delta_{1n}^{+}f_{nj}, \tag{5.70}$$

$$a^+_{nkj} = \Delta^-_n\left[\lambda^{1-2\alpha}f^+_{nj}\delta_{k1} + \sum_{p=1}^{k}\left(\frac{a^-_{p,k+1-p,j}}{2n+2p-2} + \frac{b_{p,k+1-p,j}}{2n+2p-1-2\alpha}\right)\right], \quad k=1,2,\ldots, \tag{5.71}$$

$$a^-_{nkj} = -\Delta^+_{0n}\sum_{p=1}^{k-1}\frac{a^+_{p,k-p,j}}{2n+2p-2}, \quad b_{nkj} = -\Delta^+_{1n}\sum_{p=1}^{k-1}\frac{a^+_{p,k-p,j}}{2n+2p-1-2\alpha}, \quad k=2,3,\ldots, \tag{5.72}$$

where δ_{k1} is Kronecker's symbol. Formulae (5.68)–(5.72) yield full, rapidly converging asymptotic expansions on λ for the coefficients $A^\pm_n$ and B_n. The sequence of the calculations of the elements of series (5.68) and (5.69) can be presented as follows

$$\left\{a^-_{n1j}, b_{n1j}\right\} \Longrightarrow \left\{a^+_{n1j}, a^-_{n2j}, b_{n2j}\right\} \Longrightarrow \left\{a^+_{n2j}, a^-_{n3j}, b_{n3j}\right\} \Longrightarrow \cdots, \quad n=1,2\ldots; \quad j=1,2. \tag{5.73}$$

Now define the constant C. From additional condition (5.10), as a consequence of (5.14) we deduce

$$\Phi^-_1(0) = -\frac{\pi}{2a_0}, \tag{5.74}$$

On the other hand, taking the limit as $s \to 0$ we obtain from (5.52)

$$-\Omega^-_2(0) + C + \Psi^+_0(0) = -\frac{\pi}{2a_0}K^-_0(0). \tag{5.75}$$

Taking into account (5.59) we have

$$C = \frac{1}{1+\sigma_0}\left[\Omega^-_2(0) - \sigma_1 - \frac{\pi}{2a_0\Gamma(\alpha)}\right], \quad \sigma_j = \sum_{m=1}^{\infty}\frac{A^+_{mj}}{2\alpha-2m}. \tag{5.76}$$

The definition of the constant C completes the solution of the 2×2 matrix Wiener–Hopf problem. Applying inverse Mellin transforms to formulae (5.51) and (5.52) we obtain the solution of the system of integral eqns (5.5) and (5.6)

$$\chi_1(t) = \frac{1}{2\pi i}\int_\Gamma \Phi^-_1(s)\left(\frac{t}{a_0}\right)^{-s-1}\mathrm{d}s, \quad 0<t<a_0, \tag{5.77}$$

$$\chi_2(t) = \frac{1}{2\pi i}\int_\Gamma \Phi^-_2(s)\left(\frac{t}{b_0}\right)^{-s-1}\mathrm{d}s, \quad 0<t<b_0. \tag{5.78}$$

These integrals will be calculated in the next section.

In conclusion, we give a summary of the main steps of the algorithm of the solution of system (5.5) and (5.6) that can be applied to a system of the Mellin convolution-type equations on different segments $(0, a_0)$, $(0, b_0)$

$$a_{11}\chi_1(t) + a_{12}\chi_2(t) + \int_0^{a_0} l_{11}\left(\frac{t}{\tau}\right)\chi_1(\tau)\frac{\mathrm{d}\tau}{\tau} + \int_0^{b_0} l_{12}\left(\frac{t}{\tau}\right)\chi_2(\tau)\frac{\mathrm{d}\tau}{\tau} = f_1(t), \quad 0<t<a_0, \tag{5.79}$$

$$a_{22}\chi_1(t) + a_{22}\chi_2(t) + \int_0^{a_0} l_{21}\left(\frac{t}{\tau}\right)\chi_1(\tau)\frac{\mathrm{d}\tau}{\tau} + \int_0^{b_0} l_{22}\left(\frac{t}{\tau}\right)\chi_2(\tau)\frac{\mathrm{d}\tau}{\tau} = f_2(t), \quad 0<t<b_0. \tag{5.80}$$

We suppose that the Mellin transforms of the functions l_{km} are meromorphic and without loss of generality that $\lambda = b_0 a_0^{-1} < 1$.

1. Extrapolation (5.11) and (5.12) of system (5.5) and (5.6) over the positive half-axis and calculation of the Mellin transforms of the kernels of system (5.11) and (5.12).
2. Transformation of the first equation in system (5.17) to form (5.23), in order to remove the product of the unknown function $\Phi_1^-(s)$ that is analytical in D^- and the term λ^{-s-1} that has an essential singularity at infinity in D^-.
3. Factorization (5.27) of the function $K_0(s)$, the function that is in the lower left-hand-side corner of the matrix **G**, either in an explicit form as (5.28) or in terms of Cauchy integrals (see Gakhov, 1966). Reduction of the system to form (5.29) and (5.30).
4. In the case of the non-homogeneous system (5.79) and (5.80) ($f_j(t) \neq 0$) replacement of the known summands in (5.29) and (5.30) by the difference of the boundary values of two functions (5.31) and (5.32) which are analytic in D^+ and D^-, respectively (using the Sokhotski–Plemelj formulae, for example).
5. Specification of the poles of functions (5.39), introduction of functions (5.40)–(5.41) and delimination of the poles via the subtraction of these functions from both sides of system (5.29) and (5.30).
6. Application of Abelian-type theorems and Liouville's theorem and definition of solution (5.51) and (5.52) that contains unknown coefficients [residues (5.42)–(5.44)].
7. Solution of the infinite algebraic system (5.60)–(5.62) either by the reduction method (5.65)–(5.67) or in terms of the recurrence relations (5.68)–(5.72). Due to the presence in matrix (5.18) of the terms λ^{-s-1} and λ^{s+1} (λ should not be equal to 1), the convergence of both methods is exponential.
8. Inversion of the Mellin transforms and definition of the solution of the system of integral equations.

6. Physical quantities

6.1. *Definition of the length of the slipping zone*

In order to solve the contact problem completely, we must find the position of the point b, the point of transition from slippage to bonding. Let us introduce the stress intensity factor

$$N_b = \lim_{x \to b-0} (b-x)^{1-\alpha} (\tau_{xy} + \mu\sigma_y)(x, 0) \tag{6.1}$$

and require that $N_b = 0$, i.e. that $\tau_{xy} + \mu\sigma_y = 0$, $x = b$, $y = 0$. Then the contact stresses τ_{xy} and σ_y will remain bounded in the vicinity of the point b. Note that the condition $N_b = 0$ is equivalent to the Galin condition (see Galin, 1945) as well as to the results of Spence (1973) and to the proof of Dundurs and Comninou (1979) of boundedness of the stresses at the point of transition from Coulomb friction zone to adhesion.

Since the function $\chi_2(t)$ belongs to the class (5.9) we may write

$$\chi_2(t) \sim N(b_0 - t)^{\alpha-1}, \quad t \to b_0 - 0, \tag{6.2}$$

where N is a constant. Taking into account formula (5.8) we immediately get

$$\tau_1(x) \sim \frac{PN}{2c\cos^2\dfrac{\beta b}{2}} \left(\tan\frac{\beta b}{2} - \tan\frac{\beta x}{2} \right)^{\alpha-1} \sim \frac{PN}{2c\cos^{2\alpha}\dfrac{\beta b}{2}} \left[\frac{\beta}{2}(b-x) \right]^{\alpha-1}, \quad x \to b - 0. \tag{6.3}$$

Using Theorem 2° (Appendix) and the second formula in (5.14) we arrive at the asymptotic equality

$$\Phi_2^-(s) \sim \frac{\Gamma(\alpha)}{s^\alpha} N b_0^{\alpha-1}, \quad s \to \infty, \quad s \in D^-. \tag{6.4}$$

On the other hand, due to (5.52) and (5.50), the behaviour of the function $\Phi_2^-(s)$ at infinity can be written as follows

$$\Phi_2^-(s) \sim \frac{1}{\gamma_* 2^{1-\alpha} s^\alpha} \left(\omega_0 - \sum_{m=1}^{\infty} A_m^- - \sum_{m=1}^{\infty} B_m \right), \quad s \to \infty, \quad s \in D^-, \tag{6.5}$$

where the constant ω_0 defines the behaviour of the known function $\Omega_1^+(s)$ at infinity

$$\Omega_1^+(s) \sim \frac{\omega_0}{s}, \quad s \to \infty, \quad s \in D^+, \tag{6.6}$$

$$\omega_0 = \begin{cases} \sum_{m=1}^{\infty} C_m^+, & b_0 > 1 \\ -\sum_{m=1}^{\infty} \left(C_m^- + D_m^- \right), & b_0 < 1 \end{cases}. \tag{6.7}$$

Therefore, the stress intensity factor N_b is defined by

$$N_b = \frac{P(b_0/\beta)^{1-\alpha}}{2c \cos^{2\alpha} \frac{\beta b}{2} \gamma_* \Gamma(\alpha)} \left[\omega_0 - \sum_{m=1}^{\infty} \left(A_m^- + B_m \right) \right]. \tag{6.8}$$

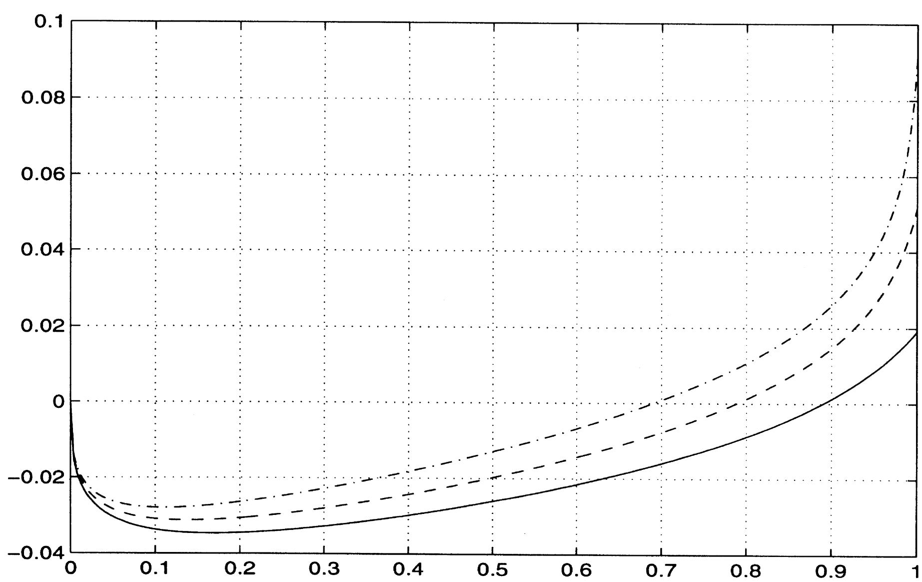

Fig. 2. Stress intensity factor N_p/P vs b(—, $c/a = 5$; - --, $c/a = 10$; · - ·, $c/a = 1000$).

We assume in all subsequent numerical examples that $\nu = 0.3$. The graphs of the stress intensity factor $P^{-1}N_b$ as a function of b for some values of c and $\mu = 0.3$ are shown in Fig. 2.

In order to have the tangential and normal contact stresses continuous at the point $x = b$, it is necessary and sufficient that $N_b = 0$ or, that it is the same as

$$\sum_{m=1}^{\infty}\left(A_m^- + B_m\right) = \omega_0. \tag{6.9}$$

The last equality is a transcendential equation for λ that is equivalent to $N_b = 0$. This equation may have a denumerable set of roots with an accumulation point at 1. According to the results of Spence (1973) only the first root gives the solution of the physical problem. We will note this root as λ. Due to $\lambda = b_0 a_0^{-1}$ and formula (5.1), the quantity b that defines the length of the zone of slip is defined by the relationship

$$b = \frac{2c}{\pi}\tan^{-1}\left(\lambda \tan\frac{\pi a}{2c}\right). \tag{6.10}$$

The normalised length of the adhesion zone $2b/a$ depends only on Poisson's ratio ν, the coefficient of friction μ and the parameter $(c-a)/a$, where a is the length of the punches and $2(c-a)$ is the distance between the stamps. The dependence of the normalised half-length $\Lambda = b/a$ of the adhesion zone on the normalised distance $(c-a)/a$ between stamps for the cases $\mu = 0.1$, 0.2 and 0.3, is graphically presented in Fig. 3. As it can be seen from the graphs, the adhesion zones increase as the stamps approach each other. On the other hand, if $c \to \infty$ then $\Lambda \to \Lambda_*$, where Λ_* is the corresponding parameter for the problem for a single stamp. For example, for $\mu = 0.3$ and $(c-a)/a = 5$, 10 and 1000 we have the following values of Λ:

0.8272, 0.7731, 0.6901,

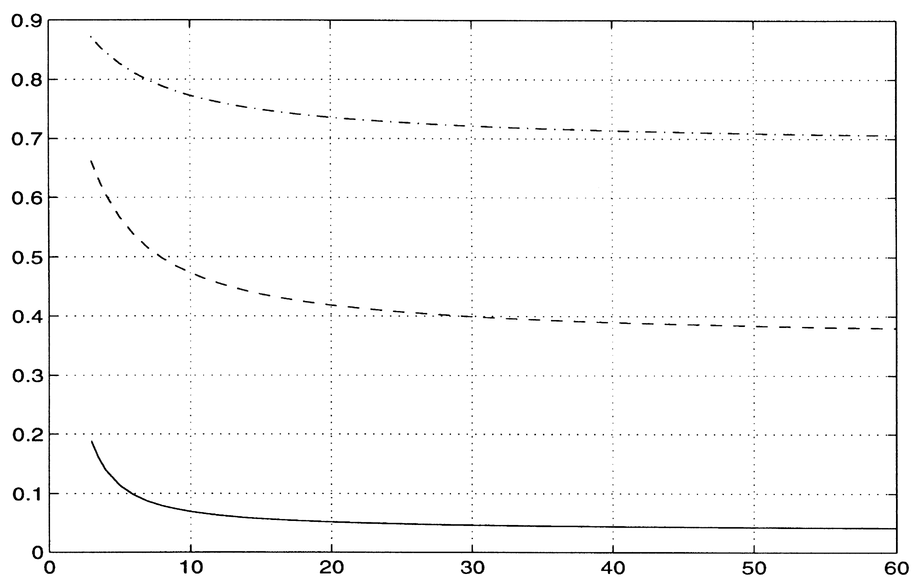

Fig. 3. Dependence of the half-length b/a of the adhesion zone on $(c-a)/a$ (—, $\mu = 0.1$; - - -, $\mu = 0.2$; · - ·, $\mu = 0.3$).

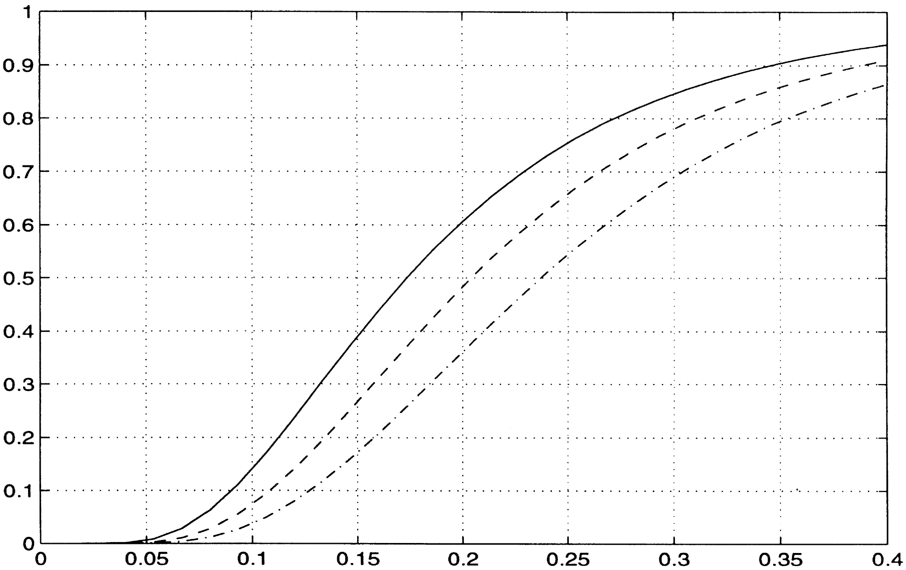

Fig. 4. Dependence of the half-length b/a of the adhesion zone on the friction coefficient μ (—, $c/a=5$; - - -, $c/a=10$; · - ·, $c/a=1000$).

respectively. In the case $c=\infty$ for $\mu=0.3$ the value of the length of the adhesion zone is known: $\Lambda_*=0.695$ (Galin, 1945). Thus, the length of the adhesion zone does not coincide with the corresponding value which produces the homogeneous system for the problem on a stamp with a horizontal base. That is similar to the problem on a stamp with non-plane profile pressed onto a half-plane under conditions of friction and adhesion (Antipov and Arutyunyan, 1992): a right part of the basic system of integral eqns (5.5) and (5.6) influences on the sliding zone length.

The dependence of the length of the adhesion zone on the friction coefficient μ for the cases $c=5$, 10 and 1000 is shown in Fig. 4. As we can observe in this figure, the curvature of the graphs increases with a decrease of this distance c.

6.2. Stress intensity factor at the edge point

If b is selected as (6.10) then it follows from (5.51) and (5.52) that the solution of the system of the functional equations has asymptotics

$$\Phi_1^+(s)=O(s^{-\alpha-1}),\quad \Phi_2^+(s)=O(s^{\alpha-1}),\quad s\to\infty,\quad s\in D^+, \tag{6.11}$$

$$\Phi_1^-(s)=O(s^{\alpha-1}),\quad \Phi_2^-(s)=O(s^{-\alpha-1}),\quad s\to\infty,\quad s\in D^-. \tag{6.12}$$

Applying Theorems 2° and 3° (Appendix) we find the asymptotics of the contact stresses and the tangential derivatives of the displacements in the neighbourhood of the singular points

$$\tau_{xy}(x,0)=O\{(a-x)^{-\alpha}\},\quad \sigma_y(x,0)=O\{(a-x)^{-\alpha}\},\quad x\to a-0, \tag{6.13}$$

$$(\tau_{xy}+\mu\sigma y)(x,0)=O\{(b-x)^{\alpha}\},\quad x\to b-0, \tag{6.14}$$

$$\frac{\partial u}{\partial x}(x,0)=O\{(x-b)^{\alpha}\},\quad x\to b+0,\quad \frac{\partial v}{\partial x}(x,0)=O\{(x-a)^{-\alpha}\},\quad x\to a+0. \tag{6.15}$$

At the point $x=b$ the contact stresses are continuous and their first derivatives have a power singularity. The normal stress intensity factor at the edge point

$$K_a=\lim_{x\to a-0}(a-x)^{\alpha}\sigma_y(x,0) \tag{6.16}$$

is computed similarly to (6.1). Since the function $\chi_1(t)$ behaves at the end point as

$$\chi_1(t)\sim M(a_0-t)^{-\alpha},\quad t\to a_0-0,\quad M=\text{const}, \tag{6.17}$$

for the normal stress $\sigma(x)$ we can write that

$$\sigma(x)\sim\frac{PM}{2c\cos^{2-2\alpha}\dfrac{\beta a}{2}}\left[\frac{\beta}{2}(a-x)\right]^{-\alpha},\quad x\to a-0. \tag{6.18}$$

If we take into account the asymptotics of the function $\Phi_1^-(s)$

$$\Phi_1^-(s)\sim\frac{\Gamma(1-\alpha)}{s^{1-\alpha}}Ma_0^{-\alpha},\quad s\to\infty,\quad s\in D^-, \tag{6.19}$$

$$\Phi_1^-(s)\sim C\left(\frac{s}{2}\right)^{\alpha-1},\quad s\to\infty,\quad s\in D^-, \tag{6.20}$$

that follows from Theorem 2° (Appendix) and (5.52), respectively, then find the constant M

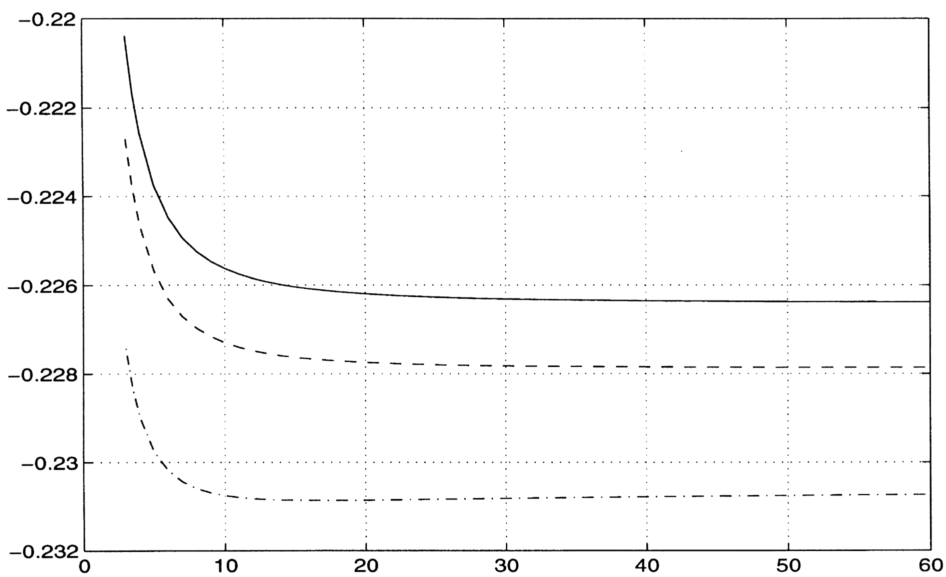

Fig. 5. Stress intensity factor K_a/P vs $(c-a)/a$ ($a=1$; —, $\mu=0.1$; - - -, $\mu=0.2$; · - ·, $\mu=0.3$).

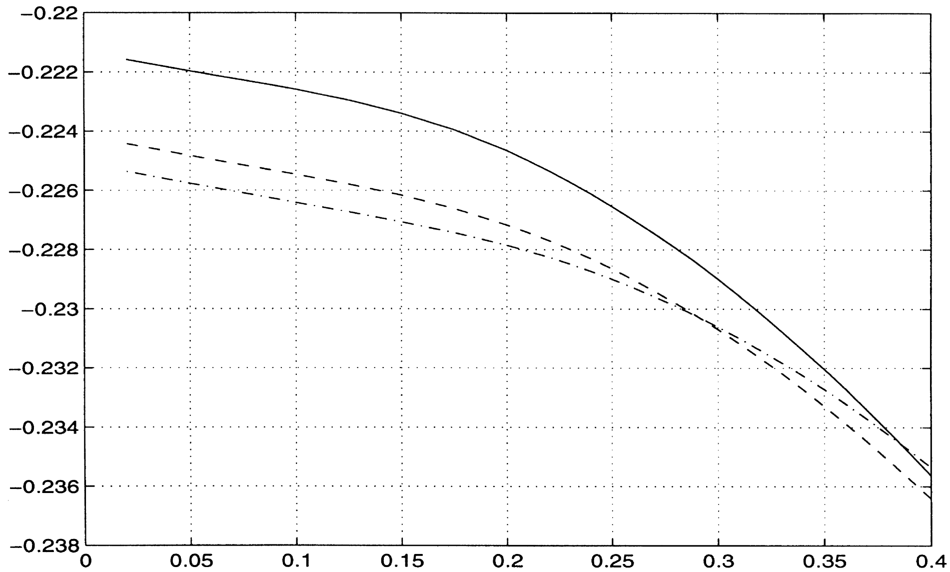

Fig. 6. Stress intensity factor K_a/P vs μ ($a=1$; —, $c=5$; - - -, $c=10$; · - ·, $c=1000$).

$$M=\frac{2^{1-\alpha}a_0^{\alpha}C}{\Gamma(1-\alpha)} \tag{6.21}$$

and therefore, from (6.16) and (6.18) we get

$$K_a=\frac{PCa_0^{\alpha}}{c\beta^{\alpha}\Gamma(1-\alpha)}\cos^{2\alpha-2}\frac{\beta a}{2}. \tag{6.22}$$

For the tangential stress intensity factor in view of Coulomb's law of dry friction, we obtain

$$L_a=\lim_{x\to a-0}(a-x)^{\alpha}\tau_{xy}(x,0)=-\mu K_a. \tag{6.23}$$

In Figs. 5 and 6 the values of the stress intensity factor K_a are presented for different values of the friction coefficient and the parameter c. As it can be seen from Fig. 5, the factor $|K_a|$ increases when the stamps approach each other. In other words, the presence of other stamps increases the stress intensity factors of both contact stresses.

6.3. Contact stresses

The contact stresses $\sigma(x)$, $\tau(x)$ are connected with the function $\chi_1(t)$, $\chi_2(x)$ by formula (5.2) and (3.22). Let us find the function $\chi_1(t)$. According to formulae (5.77) and (5.52) we have

$$\chi_1(t)=I_1(t)+\frac{\gamma}{\gamma_*}I_2(t), \tag{6.24}$$

$$I_1(t)=\frac{1}{2\pi i}\int_{\Gamma}\frac{C+\Psi_0^+(s)-\Omega_2^-(s)}{K_0^-(s)}\left(\frac{t}{a_0}\right)^{-s-1}\mathrm{d}s, \tag{6.25}$$

$$I_2(t) = \frac{1}{2\pi i}\int_\Gamma \frac{\Omega_1^-(s) - \Psi_0^-(s) - \Psi_1^-(s)}{K_0^+(s)}\left(\frac{t}{b_0}\right)^{-s-1} \mathrm{d}s. \tag{6.26}$$

Taking into account formulae (5.32) and (5.27) we get

$$I_1(t) = I_1^{(1)}(t) + I_1^{(2)}(t), \tag{6.27}$$

$$I_1^{(1)}(t) = -\frac{1}{4\kappa_0 i}\int_\Gamma \frac{t^{-s-1}\,\mathrm{d}s}{\sin\pi\left(\dfrac{s}{2}+\alpha\right)}, \tag{6.28}$$

$$I_1^{(2)}(t) = \frac{1}{2\pi i}\int_\Gamma \frac{K_0^+(s) + \left[C + \Psi_0^+(s) - \Omega_2^+(s)\right]}{K_0(s)}\left(\frac{t}{a_0}\right)^{-s-1} \mathrm{d}s. \tag{6.29}$$

The first integral $I_1^{(1)}(t)$ can be calculated by direct application of Cauchy's theorem

$$I_1^{(1)}(t) = -\frac{t^{2\alpha-1}}{\kappa_0(1+t^2)}. \tag{6.30}$$

The integrand in formula (6.29) has poles in D^+ at the points $s = -2\alpha - 2m + 2$ $(m = 1, 2, \ldots)$. Using the theory of residues we have

$$I_1^{(2)}(t) = \frac{2}{\pi\kappa_0}\sum_{m=1}^{\infty}\Gamma_{0m}\left[C - \Omega_2^+(2-2\alpha-2m) + \sum_{j=1}^{\infty}\frac{A_j^+}{-2m-2j+2}\right]\left(\frac{t}{a_0}\right)^{2\alpha+2m-3}, \tag{6.31}$$

where

$$\Gamma_{0m} = \frac{\Gamma(m-1+\alpha)}{\Gamma(m)}. \tag{6.32}$$

According to formula (5.54) and to (6.30) we transform expression (6.27) to the form

$$I_1(t) = -\frac{1}{\kappa_0}\left[\frac{t^{2\alpha-1}}{1+t^2} + \frac{\gamma}{\gamma_*}\sum_{m=1}^{\infty}\frac{A_m^-}{\Gamma_{0m}}\left(\frac{t}{b_0}\right)^{2\alpha+2m-3}\right]. \tag{6.33}$$

In order to calculate the integral $I_2(t)$, we consider two cases: $0 < t < b_0$ and $b_0 < t < a_0$. On using (5.31), in the first case we have

$$I_2(t) = -\frac{\pi\mu\gamma_*}{2\kappa_0} I_2^{(1)}(t) + I_2^{(2)}(t), \tag{6.34}$$

where

$$I_2^{(1)}(t) = \frac{1}{2\pi i}\int_\Gamma \frac{\tan\dfrac{\pi s}{2}\, t^{-s-1}\,\mathrm{d}s}{\sin\pi\left(\dfrac{s}{2}+\alpha\right)}, \tag{6.35}$$

$$I_2^{(2)}(t) = \frac{1}{2\pi i}\int_\Gamma \frac{1}{K_0^+(s)}\left(\frac{t}{b_0}\right)^{-s-1}\left[\Omega_1^+(s) - \sum_{j=1}^{\infty}\frac{A_j^-}{s+2\alpha+2j-2} - \sum_{j=1}^{\infty}\frac{B_j}{s+2j-1}\right]\mathrm{d}s, \quad 0 < t < b_0. \tag{6.36}$$

By a similar way

$$I_2^{(1)}(t) = \frac{2\left(\kappa_0 - t^{2\alpha-1}\right)}{\pi\mu\gamma(1+t^2)}, \tag{6.37}$$

$$I_2^{(2)}(t) = -\sum_{j=1}^{\infty}\left[A_j^-\frac{(t/b_0)^{2\alpha+2j-3}}{K_0^+(-2\alpha-2j+2)} + B_j\frac{(t/b_0)^{2j-2}}{K_0^+(-2j+1)}\right]. \tag{6.38}$$

Thus,

$$I_2(t) = -\frac{\gamma_*}{\gamma(1+t^2)}\left(1 - \frac{t^{2\alpha-1}}{\kappa_0}\right) + \frac{1}{\kappa_0}\sum_{j=1}^{\infty}\left[\frac{A_j^-}{\Gamma_{0j}}\left(\frac{t}{b_0}\right)^{2\alpha+2j-3} + \frac{B_j}{\Gamma_{1j}}\left(\frac{t}{b_0}\right)^{2j-2}\right], \tag{6.39}$$

$$0 < t < b_0, \quad \Gamma_{1j} = \frac{\Gamma\left(j-\frac{1}{2}\right)}{\Gamma\left(j-\alpha+\frac{1}{2}\right)}. \tag{6.40}$$

In the case $t > b_0$ we have

$$I_2(t) = \frac{2}{\pi\kappa_0^2}\sum_{m=1}^{\infty}\Gamma_m\left[\Omega_1^-(2m-2\alpha) - \sum_{j=1}^{\infty}\frac{A_j^-}{2m+2j-2} - \sum_{j=1}^{\infty}\frac{B_j}{2m+2j-2\alpha-1}\right]\left(\frac{t}{b_0}\right)^{2\alpha-2m-1}, \tag{6.41}$$

$$\Gamma_m = \frac{\Gamma(m+1-\alpha)}{\Gamma(m)}. \tag{6.42}$$

The expression in the square brackets in (6.41) can be simplified with the help of (5.56) and then

$$I_2(t) = \frac{\gamma_*}{\gamma}\sum_{m=1}^{\infty}\frac{A_m^+}{\Gamma_m}\left(\frac{t}{a_0}\right)^{2\alpha-2m-1}, \quad b_0 < t < a_0. \tag{6.43}$$

Substituting (6.33), (6.39) and (6.43) into (6.24) we arrive at the formulae

$$\chi_1(t) = -\frac{1}{1+t^2} + \frac{\gamma}{\kappa_0\gamma_*}\sum_{m=1}^{\infty}\frac{B_m}{\Gamma_{1m}}\left(\frac{t}{b_0}\right)^{2m-2}, \quad 0 < t < b_0, \tag{6.44}$$

$$\chi_1(t) = -\frac{t^{2\alpha-1}}{\kappa_0(1+t^2)} + R_1(t) - R_2(t), \quad b_0 < t < a_0, \tag{6.45}$$

$$R_1(t) = \sum_{m=1}^{\infty}\frac{A_m^+}{\Gamma_m}\left(\frac{t}{a_0}\right)^{2\alpha-2m-1}, \quad R_2(t) = \frac{\gamma}{\kappa_0\gamma_*}\sum_{m=1}^{\infty}\frac{A_m^-}{\Gamma_{0m}}\left(\frac{t}{b_0}\right)^{2\alpha+2m-3}. \tag{6.46}$$

The contact stress $\sigma(x)$ is given by

$$\sigma(x)=\frac{P}{2c\cos^2\dfrac{\beta x}{2}}\chi_1\left(\tan\frac{\beta x}{2}\right),\quad 0<x<a. \tag{6.47}$$

To calculate the tangential stresses we use formulae

$$\tau(x)=\tau_1(x)-\mu\sigma(x),\quad 0<x<b,\quad \tau(x)=-\mu\sigma(x),\quad b<x<a. \tag{6.48}$$

Therefore, we have already found the function $\tau(x)$ on the sliding zone. Let us consider the segment $0<x<b$. From (5.2) and (6.48) we have

$$\tau(x)=\frac{P}{2c\cos^2\dfrac{\beta x}{2}}\left[\chi_2\left(\tan\frac{\beta x}{2}\right)-\mu\chi_1\left(\tan\frac{\beta x}{2}\right)\right],\quad 0<x<b. \tag{6.49}$$

We substitute formulae (5.52) and (5.53) into (5.78) and find

$$\chi_2(t)=\frac{1}{2\pi i\gamma_*}\int_\Gamma K_0^-(s)\left[\Omega_1^-(s)-\Psi_0^-(s)-\Psi_1^-(s)\right]\left(\frac{t}{b_0}\right)^{-s-1}\mathrm{d}s. \tag{6.50}$$

As in the case of the function $\chi_1(t)$, we continue analytically the function $\Omega_1^-(s)$ in the half-plane D^+ and from (5.31) we get

$$\begin{aligned}\chi_2(t)=&-\frac{\mu}{4i}\int_\Gamma\frac{t^{-s-1}\,\mathrm{d}s}{\cos\dfrac{\pi s}{2}}+\frac{1}{2\pi i\gamma_*}\int_\Gamma\frac{\cot\dfrac{\pi s}{2}+\cot\pi\alpha}{K_0^+(s)}\\&\times\left[\Omega_1^+(s)-\sum_{j=1}^\infty\frac{A_j^-}{s+2\alpha+2j-2}-\sum_{j=1}^\infty\frac{B_j}{s+2j-1}\right]\left(\frac{t}{b_0}\right)^{-s-1}\mathrm{d}s.\end{aligned} \tag{6.51}$$

Application of the theory of residues yields us

$$\begin{aligned}\chi_2(t)=&-\frac{\mu}{1+t^2}-\frac{2}{\pi\kappa_0\gamma_*}\sum_{m=1}^\infty\Gamma_m\left(\frac{t}{b_0}\right)^{2m-1}\Omega_1^+(-2m)+\frac{\mu\gamma}{\gamma_*\kappa_0}\sum_{m=1}^\infty\frac{B_m}{\Gamma_{1m}}\left(\frac{t}{b_0}\right)^{2m-2}\\&+\frac{2}{\pi\gamma_*\kappa_0}\sum_{m=1}^\infty\Gamma_m\sum_{j=1}^\infty\left(\frac{A_j^-}{2\alpha+2j-2m-2}+\frac{B_j}{2j-2m-1}\right)\left(\frac{t}{b_0}\right)^{2m-1},\end{aligned} \tag{6.52}$$

where

$$\Omega_1^+(-2m)=-\frac{\mu\gamma_*}{2}\sum_{j=1}^\infty\frac{(-1)^j b_0^{-2j}\Gamma_{2j}}{m+j-\frac12},\quad b_0>1, \tag{6.53}$$

$$\Omega_1^+(-2m)=\frac{\gamma_*}{2\gamma}\sum_{j=1}^\infty(-1)^j b_0^{2j-2}\left[\frac{\kappa_0\Gamma_{1j}}{m-j+\frac12}-\frac{b_0^{2\alpha-1}\Gamma_{0j}}{2(m-j-\alpha+1)}\right],\quad b_0<1, \tag{6.54}$$

$$\Gamma_{2j} = \frac{\Gamma\left(\alpha + j - \frac{1}{2}\right)}{\Gamma\left(j + \frac{1}{2}\right)}. \tag{6.55}$$

Now, using formulae (5.44) and (6.52) we arrive at the expression for the function $\chi_2(t) - \mu\chi_1(t)$

$$\chi_2(t) - \mu\chi(t) = -\frac{2}{\pi\kappa_0\gamma_*}\sum_{m=1}^{\infty}\Gamma_m\Omega_1^{+}(-2m)\left(\frac{t}{b_0}\right)^{2m-1}$$
$$+\frac{1}{\pi\kappa_0\gamma_*}\sum_{m=1}^{\infty}\Gamma_m\left(\frac{t}{b_0}\right)^{2m-1}\sum_{j=1}^{\infty}\left(\frac{A_j^-}{\alpha+j-m-1}+\frac{B_j}{j-m-\frac{1}{2}}\right), \quad 0 < t < b_0. \tag{6.56}$$

The stresses $\tau(s)$ in the adhesion area are specified by

$$\tau(x) = \frac{P}{2c\cos^2\dfrac{\beta x}{2}}\left[\chi_2\left(\tan\frac{\beta x}{2}\right) - \mu\chi_1\left(\tan\frac{\beta x}{2}\right)\right], \quad 0 < x < b. \tag{6.57}$$

6.4. Behaviour of the contact stresses in the neighbourhood of the point b

Taking into account asymptotics (5.49) of the coefficients $A_m^{\pm}$, B_m we notice that as $t \to b_0 + 0$ the series $R_1(t)$ converges slowly. In order to study the behaviour of the function $\sigma(x)$ in the right vicinity of the point $t = b$, we use (5.56) and express the coefficients A_m^+ in (6.46) via the coefficients A_j^-, B_j. Then we change the order of summation and get

$$R_1(t) = \frac{t^{2\alpha-1}}{\kappa_0(t^2+1)} - \frac{\gamma\Gamma(2-\alpha)}{\pi\gamma_*\kappa_0^2}\left(\frac{t}{b_0}\right)^{2\alpha-3} \tag{6.58}$$

$$\times\sum_{j=1}^{\infty}\left[\frac{A_j^- F_{1j}(t)}{j} + \frac{B_j F_{2j}(t)}{j-\alpha+\frac{1}{2}} - \frac{C_j^+ F_{3j}(t)}{\frac{3}{2}-j-\alpha}\Gamma_{2j}F_{3j}(t)\right], \quad b_0 > 1, \tag{6.59}$$

$$R_1(t) = -\frac{\gamma\Gamma(2-\alpha)}{\pi\gamma_*\kappa_0^2}\left(\frac{t}{b_0}\right)^{2\alpha-3}\sum_{j=1}^{\infty}\left[\left(A_j^- + D_j^-\right)\frac{F_{1j}(t)}{j} + \left(B_j + C_j^-\right)\frac{F_{2j}(t)}{j-\alpha+\frac{1}{2}}\right], \quad b_0 < 1,$$

where

$$F_{1j}(t) = F\left(2-a, j; j+1; \frac{b_0^2}{t^2}\right), \tag{6.60}$$

$$F_{2j}(t) = F\left(2-\alpha, j-\alpha+\frac{1}{2}; j-\alpha+\frac{3}{2}; \frac{b_0^2}{t^2}\right), \tag{6.61}$$

$$F_{3j}(t) = F\left(2-\alpha, \frac{3}{2}-j-\alpha; \frac{5}{2}-j-\alpha; \frac{b_0^2}{t^2}\right), \tag{6.62}$$

$F(a, b; c; x)$ is a Gaussian hypergeometric function. We introduce the function

$$\mathfrak{F}(\nu;\,t)=F\left(2-\beta,\nu;\nu+1;\,\frac{b_0^2}{t^2}\right) \tag{6.63}$$

and express the functions $F_{kj}(t)$ in terms of this function

$$F_{1j}(t)=\mathfrak{F}(j;\,t),\quad F_{2j}(t)=\mathfrak{F}\big(j-\alpha+\tfrac{1}{2};\,t\big),\quad F_{3j}(t)=\mathfrak{F}\big(-j-\alpha+\tfrac{3}{2};\,t\big). \tag{6.64}$$

The function $\mathfrak{F}(\nu;\,t)$ admits the following presentation in the right neighbourhood of the point $t=b_0$ [see formula 9.131(2), Gradshtein and Ryzhik, 1965]

$$\mathfrak{F}(\nu;\,t)=\frac{\Gamma(\nu+1)\Gamma(\alpha-1)}{\Gamma(\nu+\alpha-1)}\left(\frac{t}{bo}\right)^{2\nu}+\frac{\nu}{1-\alpha}\left(1-\frac{b_0^2}{t^2}\right)^{\alpha-1}+\nu\Gamma(\alpha-1)\mathfrak{F}^*(t), \tag{6.65}$$

where

$$\mathfrak{F}^*(\nu;\,t)=-\frac{1}{\Gamma(\alpha)}\sum_{j=1}^{\infty}\frac{(\nu-1+\alpha)_j}{(\alpha)_j}\left(1-\frac{b_0^2}{t^2}\right)^{j+\alpha-1}. \tag{6.66}$$

At the first sight, the function $R_1(t)$ has a power singularity as $t\to b_0+0$. But if we take into account that for the root of eqn (6.9)

$$\sum_{j=1}^{\infty}\left(A_j^-+B_j-C_j^+\right)=0,\quad b_0>1 \tag{6.67}$$

and

$$\sum_{j=1}^{\infty}\left(A_j^-+B_j+C_j^-+D_j^-\right)=0,\quad b_0<1, \tag{6.68}$$

we get for $b_0<t<a$

$$\begin{aligned}R_1(t)=\frac{1}{1+t^2}\left(\frac{t^{2\alpha-1}}{\kappa_0}-1\right)+\frac{\gamma}{\kappa_0\gamma_*}\left(\frac{t}{b_0}\right)^{2\alpha-3}\sum_{j=1}^{\infty}\Bigg\{A_j^-\left[\frac{1}{\Gamma_{0j}}\left(\frac{t}{b_0}\right)^{2j}+\mathfrak{F}^*(j;\,t)\right]\\+B_j\left[\frac{1}{\Gamma_{1j}}\left(\frac{t}{b_0}\right)^{2j-2\alpha+1}+\mathfrak{F}^*\left(j-\alpha+\frac{1}{2};\,t\right)\right]+W_j(t)\Bigg\},\end{aligned} \tag{6.69}$$

where

$$W_j(t)=-C_j^+\mathfrak{F}^*\big(\tfrac{3}{2}-j-\alpha;\,t\big),\quad b_0>1, \tag{6.70}$$

$$W_j(t)=D_j^-\mathfrak{F}^*(j;\,t)+C_j^-\mathfrak{F}^*\big(j+\tfrac{1}{2}-\alpha;\,t\big),\quad b_0<1. \tag{6.71}$$

Thus, we arrive at boundedness of the function $\chi_1(t)$ at the point b_0. Hence, the contact stresses $\sigma(x)$ are bounded at the point b. For the sake of completeness we have to prove that the stresses $\sigma(x)$ are continuous at the point of transition from slippage to bonding. Since $\mathfrak{F}^*(\nu;\,b_0)=0$ the limit expression $R_1(b_0+0)$ can be written as follows

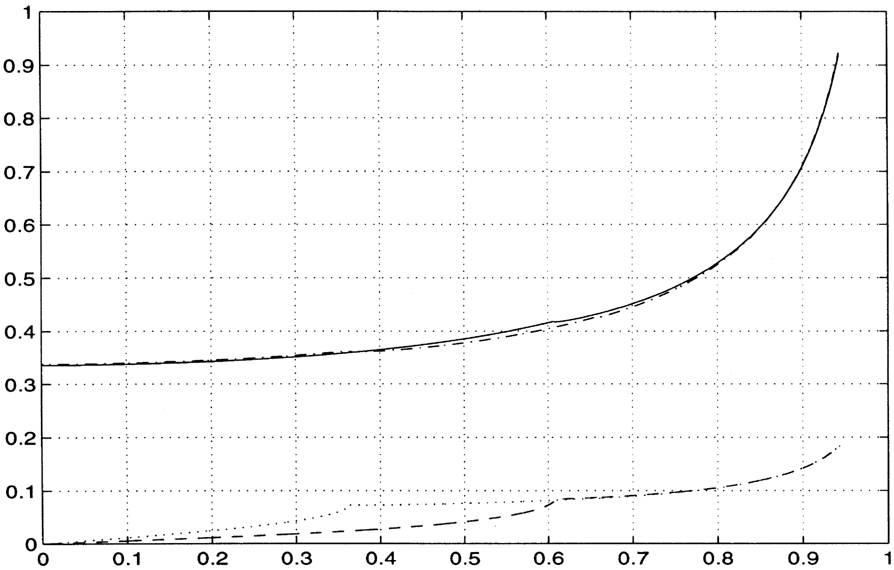

Fig. 7. Distribution of the contact stresses along the contact zone ($0 < x < a$) for the following parameters: $a = 1$, $\mu = 0.2$ (—, $-P^{-1}\sigma_y(x, 0)$, $c = 5$; $\cdot - \cdot$, $-P^{-1}\sigma_y(x, 0)$, $c = 1000$; - - -, $P^{-1}\tau_{xy}(x, 0)$, $c = 5$; $\cdots$, $P^{-1}\tau_{xy}(x, 0)$, $c = 1000$).

$$R_1(b_0 + 0) = \frac{b_0^{2\alpha-1}}{\kappa_0\left(b_0^2 + 1\right)} - \frac{1}{b_0^2 + 1} + \frac{\gamma}{\gamma_*\kappa_0}\sum_{j=1}^{\infty}\left(\frac{A_j^-}{\Gamma_{0j}} + \frac{B_j}{\Gamma_{1j}}\right) \tag{6.72}$$

in both cases $b_0 > 1$ and $b_0 < 1$. Substituting this formula into (6.45) and comparing it with (6.44) we get

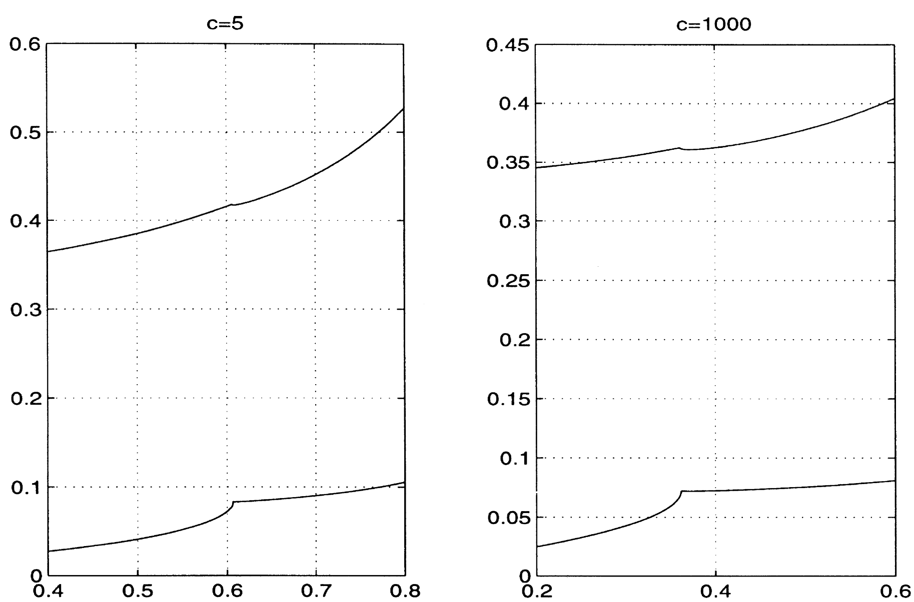

Fig. 8. The contact stresses in the neighbourhood of the point b ($a = 1$, $\mu = 0.2$).

$$\chi_1(b_0+0)=\chi_1(b_0-0)=-\frac{1}{1+b_0^2}+\frac{\gamma}{\kappa_0\gamma_*}\sum_{m=1}^{\infty}\frac{B_m}{\Gamma_{1m}}, \tag{6.73}$$

$$\frac{B_m}{\Gamma_{1m}}=O\left(m^{\alpha-1}\lambda^{2m}\right),\quad m\to\infty\quad (0<\lambda<1). \tag{6.74}$$

In a similar manner, it may be shown, that the tangential stresses are continuous at the point b. The derivatives with respect to x of the stresses $\sigma_y(x,0)$ and $\tau_{xy}(x,0)$ have a power singularity at the point b.

In Fig. 7 the graphs of the normal and tangential stresses for the cases $c = 5$ and $c = 1000$ when $\mu = 0.2$ are shown. These plots are in good agreement with the corresponding curves of Spence (1973): the graphs of stresses have a salient point under transition from bonding to slippage. It should be noted that the analytical formulae (6.44)–(6.46), (6.69) and (6.73) allow us to calculate the values of stresses with high exactness at the salient point b. The scaled graphs of the contact stresses $-P^{-1}\sigma(x)$ and $P^{-1}\tau(x)$ in the neighbourhood of the point b for the same parameters as in Fig. 7, are presented in Fig. 8.

7. Conclusion

We have solved analytically the contact problem for a periodic system of stamps with friction and adhesion. The problem was reduced to a system of two singular integral equations with Hilbert's kernels and then to a 2×2 matrix Wiener–Hopf problem that was solved efficiently. It was shown that this technique admits generalization for a system of two Mellin convolution-type equations on different segments if Mellin transforms of the kernels are meromorphic functions. Dependence of the sliding zone length on the distance between stamps and the friction coefficient was studied. The length of the adhesion area and the stress intensity factor at the edge point increase when the stamps approach each other. It has been shown that the contact stresses are bounded and continuous at the point of transition from slippage to bonding.

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Appendix: the basic theorems

Theorem 1° (the Mellin convolution theorem, Titchmarsh, 1948). Let $x^{\kappa}f(x)\in L(0,\infty)$ and $x^{\kappa}g(x)\in L(0,\infty)$ and let

$$h(x)=\int_0^{\infty} f(y)g\left(\frac{x}{y}\right)\frac{\mathrm{d}y}{y}, \tag{A.1}$$

then $x^{\kappa}h(x)\in L(0,\infty)$ and its Mellin transform $H(s)$ is equal to $F(s)G(s)$ with $\Re(s)=\kappa+1$ where

$$\|H(s), G(s), F(s)\| = \int_0^{\infty} \|h(y), g(y), f(y)\| x^{s-1}\,\mathrm{d}x. \tag{A.2}$$

Theorem 2° (the first Abelian theorem, Doutsch, 1950). Let

$$f(x) \sim A(1-x)^{\alpha-1}, \quad x \to 1-0, \tag{A.3}$$

where A = const, $\Re(\alpha) > 0$ and the integral

$$F^{-}(s) = \int_0^1 f(x) x^s\,\mathrm{d}x \tag{A.4}$$

is an absolutely convergent function in the half-plane $\Re(s) > \kappa$. Then the behaviour of the function $F^{-}(s)$ at infinity in this half-plane is defined by

$$F^{-}(s) \sim \frac{A\Gamma(\alpha)}{s^{\alpha}}, \quad s \to \infty, \quad |\arg(s)| \leqslant \phi_0 < \frac{\pi}{2}. \tag{A.5}$$

Theorem 3° (the second Abelian theorem, Doutsch, 1950). Let

$$f(x) \sim A(x-1)^{\alpha-1}, \ x \to 1+0, \tag{A.6}$$

where A = const, $\Re(\alpha) > 0$ and the integral

$$F^{+}(s) = \int_1^{\infty} f(x) x^s\,\mathrm{d}x \tag{A.7}$$

is an absolutely convergent function in the half-space $\Re(s) < \kappa$. Then the behaviour of the function $F^{+}(s)$ at infinity in this half-plane is defined by

$$F^{+}(s) \sim \frac{A\Gamma(\alpha)}{(-s)^{\alpha}}, \quad s \to \infty, \quad |\arg(-s)| \leqslant \phi_0 < \frac{\pi}{2}. \tag{A.8}$$

Theorem 4° (Liouville's theorem, Gakhov, 1966). Let the function $F(s)$ be analytic in the entire plane of the complex variable, except at infinity, where it has a pole and suppose that the principal part of the expansion of the function $F(s)$ at infinity has the form

$$C_0 + C_1 s + C s^2 + \cdots + C_{\kappa} s^{\kappa}.$$

Then the function $F(s)$ is the polynomial of degree κ: $F(s) = C_0 + C_1 s + \cdots + C_{\kappa} s^{\kappa}$ everywhere in the plane. In particular, if the function $F(s)$ is a constant at infinity then it is that constant in the entire plane.

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